DEFORMATIONS OF $\mathcal{W}$–ALGEBRAS ASSOCIATED TO SIMPLE LIE ALGEBRAS

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Abstract. Deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ associated to an arbitrary simple Lie algebra $\mathfrak{g}$ is defined together with its free field realizations and the screening operators. Explicit formulas are given for generators of $\mathcal{W}_{q,t}(\mathfrak{g})$ when $\mathfrak{g}$ is of classical type. These formulas exhibit a deep connection between $\mathcal{W}_{q,t}(\mathfrak{g})$ and the analytic Bethe Ansatz in integrable models associated to quantum affine algebras $U_q(\widehat{\mathfrak{g}})$ and $U_t({}^L\widehat{\mathfrak{g}})$. The scaling limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ is closely related to affine Toda field theories.

1. Introduction

In this paper we define deformations of the $\mathcal{W}$–algebras associated to arbitrary simple Lie algebras, their free field realizations and the screening operators. The deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ associated to a simple Lie algebra $\mathfrak{g}$ is a family of associative algebras depending on two parameters, q and t. In the case when $\mathfrak{g} = A_\ell$, the algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ has been constructed in [41, 15, 2] (see also [34, 17]). Various limits of $\mathcal{W}_{q,t}(\mathfrak{g})$ can be identified with previously known algebras. In particular, in the limit $q \to 1$ with $t = q^\beta$ and β fixed, we recover the ordinary $\mathcal{W}$–algebra corresponding to $\mathfrak{g}$.

In the limits $t \to 1$ and $q \to 1$ the algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ becomes commutative, and has a natural Poisson structure. We conjecture that the Poisson algebra $\mathcal{W}_{q,1}(\mathfrak{g})$ is isomorphic to the center of the quantized enveloping algebra $U_q(\widehat{\mathfrak{g}})$ at the critical level (see [17]), while the Poisson algebra $\mathcal{W}_{1,t}(\mathfrak{g})$ is isomorphic to the Poisson algebra obtained by the difference Drinfeld-Sokolov reduction of the loop group $G((z))$ (see [21, 40]). In the case $\mathfrak{g} = A_\ell$, the first conjecture follows from results of [17] and the second follows from results of [21, 40]; we have also checked the second conjecture for $\mathfrak{g} = C_2$ (see Appendix B).

Another important limit is $q \to \epsilon$, where $\epsilon = 1$ for simply-laced $\mathfrak{g}$, and $\epsilon = \exp(\pi i/r^\vee)$ for nonsimply-laced $\mathfrak{g}$. Here $r^\vee$ is the order of the automorphism of a simply-laced Lie algebra that gives rise to $\mathfrak{g}$. In this limit, $\mathcal{W}_{q,t}(\mathfrak{g})$ contains a commutative subalgebra $\mathcal{W}'_{\epsilon,t}(\mathfrak{g})$, which has a natural Poisson structure. We conjecture that the Poisson algebra $\mathcal{W}'_{\epsilon,t}(\mathfrak{g})$ is isomorphic to the center of the quantized enveloping algebra $U_t({}^L\widehat{\mathfrak{g}})$ at the critical level. Here ${}^L\widehat{\mathfrak{g}}$ is the affine algebra that is Langlands dual to $\widehat{\mathfrak{g}}$, that is the Cartan matrix of ${}^L\widehat{\mathfrak{g}}$ is the transpose of the Cartan matrix of $\widehat{\mathfrak{g}}$.

We remark that for simply-laced $\mathfrak{g}$, $\mathcal{W}_{q,t}(\mathfrak{g}) \simeq \mathcal{W}_{t,q}(\mathfrak{g})$, but for nonsimply-laced $\mathfrak{g}$, $\mathcal{W}_{q,t}(\mathfrak{g})$ is not isomorphic to $\mathcal{W}_{t,q}({}^L\mathfrak{g})$, as one would expect by analogy with Langlands duality of the ordinary $\mathcal{W}$–algebras [13]. This means that the duality becomes more complicated after deformation.

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We give explicit formulas for generators of $\mathcal{W}_{q,t}(\mathfrak{g})$ when $\mathfrak{g}$ is of classical type. These formulas exhibit a remarkable connection with the analytic Bethe Ansatz in integrable models associated with quantum affine algebras (see [5, 37, 38, 4, 29]). Recall that in [17] we conjectured (and proved in the case $\mathfrak{g} = A_\ell$) that the formulas for the free field realization of the center of $U_q(\widehat{\mathfrak{g}})$, i.e., $\mathcal{W}_{q,1}(\mathfrak{g})$, coincide with Bethe Ansatz formulas for the eigenvalues of the transfer-matrices in the $U_q(\widehat{\mathfrak{g}})$ integrable model. In this paper we will see further evidence of that. We will also see that the formulas for the free field realization of $\mathcal{W}'_{\epsilon,t}(\mathfrak{g})$ coincide with Bethe Ansatz formulas for transfer-matrices in the $U_t({}^L\widehat{\mathfrak{g}})$ model. Furthermore, the formulas for the free field realization of $\mathcal{W}_{1,t}(\mathfrak{g})$ exhibit similarity with formulas for the eigenvalues of transfer-matrices in the integrable model associated to $U_t(\widehat{\mathfrak{g}}^\vee)$, where $\widehat{\mathfrak{g}}^\vee = {}^L(\widehat{{}^L\mathfrak{g}})$.

Thus, the free field realization of $\mathcal{W}_{q,t}(\mathfrak{g})$ connects the analytic Bethe Ansatz formulas for $U_q(\widehat{\mathfrak{g}})$, $U_t({}^L\widehat{\mathfrak{g}})$ and $U_t(\widehat{\mathfrak{g}}^\vee)$.

The Bethe Ansatz formulas can be written for any finite-dimensional representation of quantum affine algebra, and can be thought of as q–analogues of the character formulas. Numerous examples are known in the literature, see [38, 29, 30]. However, there seems to be no systematic algorithm for writing these formulas in general. We hope that one can use the property of commutativity with the screening operators as the defining property for these "q–characters". This may give us some insights into the category of finite-dimensional representations of quantum affine algebras. Furthermore, it would then appear that $\mathcal{W}_{q,t}(\mathfrak{g})$ is a simultaneous deformation of representation rings of $U_q(\widehat{\mathfrak{g}})$ and $U_t({}^L\widehat{\mathfrak{g}})$. We will discuss these issues in more detail in [20].

The paper is organized as follows: in Sect. 2 we define a two-parameter family of deformations of the Cartan matrix associated to each simple Lie algebra. In Sect. 3 we define the Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ and the screening operators. We then define $\mathcal{W}_{q,t}(\mathfrak{g})$ as the centralizsr of the screening operators in $\mathcal{H}_{q,t}(\mathfrak{g})$. We conjecture the form of the generators of $\mathcal{W}_{q,t}(\mathfrak{g})$ and derive from this conjecture the exchange relations between the generators of $\mathcal{W}_{q,t}(\mathfrak{g})$.

We also compute the relations between the screening currents. It was shown in [15] that for $\mathfrak{g} = A_\ell$ they satisfy elliptic relations, which can be considered as elliptic analogues of Drinfeld's relations in $U_q(\widehat{\mathfrak{n}})$ [11]. This fact has been used in [24, 16] to construct two-sided resolutions of irreducible representations of deformed $\mathcal{W}$–algebras. Here we obtain the relations for arbitrary $\mathfrak{g}$.

In Sect. 4 we study various limits of $\mathcal{W}_{q,t}(\mathfrak{g})$ and identify them with some known algebras. In Sect. 5 we give explicit formulas for the generating field $T_1(z)$ of $\mathcal{W}_{q,t}(\mathfrak{g})$, in the case when $\mathfrak{g}$ is a simple Lie algebra of classical type. In Sect. 6 we discuss the connection between $\mathcal{W}_{q,t}(\mathfrak{g})$ and the analytic Bethe Ansatz. In Sect. 7 we define the deformed $\mathcal{W}$–algebra associated to the self-dual twisted affine algebra $A_{2\ell}^{(2)}$ generalizing the recent work [7].

In Sect. 8, we consider the scaling limit of the exchange relations between the generators of $\mathcal{W}_{q,t}(\mathfrak{g})$ and between the screening currents. We show that the scaling limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ can be identified with the Faddeev-Zamolodchikov algebra of an affine Toda field theory. In the case $\mathfrak{g} = A_\ell$ this has been suggested by S. Lukyanov [32, 33] (see also [31]). The scaling limit of our free field realization can therefore be used to obtain

explicit formulas for form-factors in general affine Toda field theories along the lines of [32, 33, 7]. On the other hand, relations between the screening currents give rise in the scaling limit to Drinfeld's relations, for both twisted and non-twisted affine algebras.

In Sect. 9 we define the analogues of vertex operators (primary fields) for $\mathcal{W}_{q,t}(\mathfrak{g})$, which correspond to the fundamental representations of $U_q(\widehat{\mathfrak{g}})$. Using these vertex operators and the screening operators, one can give complete bosonization of the vertex operators in the general solvable SOS models, along the lines of [35, 1]. We note that the last two sections are somewhat more physics oriented than the rest of the paper.

In Appendix A we give some explicit formulas for the Poisson algebras $\mathcal{H}_{q,1}(\mathfrak{g})$ and $\mathcal{W}_{q,1}(\mathfrak{g})$ for Lie algebras of classical types. In Appendix B we discuss the difference Drinfeld-Sokolov reduction in the case $\mathfrak{g} = C_2$. Appendix C contains explicit formulas for the matrices $M(q,t)$ corresponding to the classical Lie algebras. In Appendix D we recall the definition of deformed chiral algebra from [19].

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2. Two-parameter deformations of Cartan matrices

2.1. **General formula.** Let $\mathfrak{g}$ be a simple Lie algebra of rank ℓ. Let $(\cdot,\cdot)$ be the invariant inner product on $\mathfrak{g}$, normalized as in [27], so that the square of the maximal root equals 2. Let $\{\alpha_1,\dots,\alpha_\ell\}$ and $\{\omega_1,\dots,\omega_\ell\}$ be the sets of simple roots and of fundamental weights of $\mathfrak{g}$, respectively. We have:

$$(\alpha_i,\omega_j) = \frac{(\alpha_i,\alpha_i)}{2}\delta_{i,j}.$$

Let $r^\vee$ be the maximal number of edges connecting two vertices of the Dynkin diagram of $\mathfrak{g}$. Thus, $r^\vee = 1$ for simply-laced $\mathfrak{g}$, $r^\vee = 2$ for B_ℓ, C_ℓ, F_4, G_2, and $r^\vee = 3$ for D_4. Set

$$D = \operatorname{diag}(r_1,\dots,r_\ell),$$

where

$$r_i = r^\vee \frac{(\alpha_i,\alpha_i)}{2}. \tag{2.1}$$

All r_i's are integers; for simply-laced $\mathfrak{g}$, D is the identity matrix.

Now let $C = (C_{ij})_{1\le i,j\le \ell}$ be the *Cartan matrix* of $\mathfrak{g}$. We have:

$$C_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}.$$

Denote by $I = (I_{ij})_{1\le i,j\le\ell}$ the *incidence matrix*,

$$I_{ij} = 2\delta_{i,j} - C_{ij}.$$

Let $B = (B_{ij})_{1\le i,j\le\ell}$ be the following matrix:

$$B = DC,$$

i.e.,

$$B_{ij} = r^\vee(\alpha_i,\alpha_j).$$

Now let q, t be indeterminates. We will use the standard notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We define $\ell \times \ell$ matrices $C(q,t)$, $D(q,t)$, and $B(q,t)$ by the formulas

$$C_{ij}(q,t) = (q^{r_i} t^{-1} + q^{-r_i} t)\delta_{i,j} - [I_{ij}]_q, \tag{2.2}$$

$$D(q,t) = \operatorname{diag}([r_1]_q, \dots, [r_\ell]_q), \tag{2.3}$$

$$B(q,t) = D(q,t)C(q,t).$$

Thus,

$$B_{ij}(q,t) = [r_i]_q \left((q^{r_i} t^{-1} + q^{-r_i} t)\delta_{i,j} - [I_{ij}]_q \right). \tag{2.4}$$

It is easy to see that the matrix $B(q,t)$ is symmetric. For simply-laced $\mathfrak{g}$,

$$C_{ij}(q,t) = B_{ij}(q,t) = (qt^{-1} + q^{-1}t)\delta_{i,j} - I_{ij}.$$

Clearly, the limits of $C(q,t)$, $D(q,t)$, and $B(q,t)$ as $q \to 1$ and $t \to 1$ coincide with C, D, and B, respectively. Note also that

$$B_{ij}(q,1) = [B_{ij}]_q,$$

and

$$B_{ij}(1,t) = r_i((t + t^{-1})\delta_{ij} - I_{ij}).$$

It is interesting that for all $\mathfrak{g}$, the Coxeter number h enters the determinant of $C(q,t)$ as the power of t, and $r^\vee h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, enters the determinant as the power of q. For instance, for $\mathfrak{g} = A_\ell$:

$$\det C(q,t) = \frac{q^{\ell+1} t^{-\ell-1} - q^{-\ell-1} t^{\ell+1}}{qt^{-1} - q^{-1}t}$$

(in this case $h = r^\vee h^\vee = \ell + 1$), for $\mathfrak{g} = B_\ell$:

$$\det C(q,t) = q^{2\ell-1} t^{-\ell} + q^{-2\ell+1} t^{\ell}$$

(in this case $h = 2\ell, r^\vee h^\vee = 2(2\ell - 1)$), for $\mathfrak{g} = C_\ell$:

$$\det C(q,t) = q^{\ell+1} t^{-\ell} + q^{-\ell-1} t^{\ell}$$

(in this case $h = 2\ell, r^\vee h^\vee = 2(\ell + 1)$), etc.

3. Deformed $\mathcal{W}$–algebras

3.1. **Heisenberg algebra $\mathcal{H}_{q,t}(\mathfrak{g})$.** Let $\mathcal{H}_{q,t}(\mathfrak{g})$ be the Heisenberg algebra with generators $a_i[n], i = 1, \dots, \ell; n \in \mathbb{Z}$, and relations

$$[a_i[n], a_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n}) B_{ij}(q^n, t^n)\delta_{n,-m}. \tag{3.1}$$

In this and other formulas of this type we understand that the 0th generator commutes with all other generators: $[a_i[0], a_j[m]] = 0, \forall m \in \mathbb{Z}$.

The generators $a_i[n]$ are "root" type generators of $\mathcal{H}_{q,t}(\mathfrak{g})$. There is a unique set of "fundamental weight" type generators, $y_i[n], i = 1, \ldots, \ell; n \in \mathbb{Z}$, that satisfy:

$$[a_i[n], y_j[m]] = \frac{1}{n}(q^{r_i n} - q^{-r_i n})(t^n - t^{-n})\delta_{i,j}\delta_{n,-m}. \tag{3.2}$$

They have the following commutation relations:

$$[y_i[n], y_j[m]] = \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n})M_{ij}(q^n, t^n)\delta_{n,-m}, \tag{3.3}$$

where $(M_{ij}(q,t))_{1\leq i,j\leq \ell}$ is the following matrix

$$\begin{aligned} M(q,t) &= D(q,t)C(q,t)^{-1} \\ &= D(q,t)B(q,t)^{-1}D(q,t). \end{aligned} \tag{3.4}$$

Note that

$$a_j[n] = \sum_{j=1}^{\ell} C_{ij}(q^n, t^n)y_j[n]. \tag{3.5}$$

Introduce the generating series:

$$A_i(z) = t^{2(\rho^\vee,\alpha_i)}q^{-2r^\vee(\rho,\alpha_i)+2a_i[0]} : \exp\left(\sum_{m\neq 0} a_i[m]z^{-m}\right) :,$$

$$Y_i(z) = t^{2(\rho^\vee,\omega_i)}q^{-2r^\vee(\rho,\omega_i)+2y_i[0]} : \exp\left(\sum_{m\neq 0} y_i[m]z^{-m}\right) : .$$

Note that $(\rho^\vee, \alpha_i) = 1, r^\vee(\rho,\alpha_i) = r_i$.

For each weight μ of the Cartan subalgebra of $\mathfrak{g}$, let π_μ be the Fock representation of $\mathcal{H}_{q,t}(\mathfrak{g})$ generated by a vector v_μ, such that $a_i[n]v_\mu = 0, n > 0$, and $a_i[0]v_\mu = (\mu, \alpha_i)v_\mu$.

3.2. **Screening operators.** Set $t = q^\beta$. Introduce the shift operators $e^{Q_i}, i = 1, \ldots, \ell$, acting from π_μ to $\pi_{\mu+\beta\alpha_i}$, which satisfy commutation relations

$$[a_i[n], e^{Q_j}] = B_{ij}\beta\delta_{n,0}e^{Q_j}. \tag{3.6}$$

Let

$$s_i^+[m] = \frac{a_i[m]}{q^{mr_i} - q^{-mr_i}}, \quad m \neq 0, \qquad s_i^+[0] = a_i[0]/r_i, \tag{3.7}$$

$$s_i^-[m] = \frac{a_i[m]}{t^m - t^{-m}}, \quad m \neq 0, \qquad s_i^-[0] = a_i[0]/\beta. \tag{3.8}$$

Now define the *screening currents* by the formulas

$$S_i^+(z) = e^{-Q_i/r_i} z^{-s_i^+[0]} : \exp\left(\sum_{m\neq 0} s_i^+[m] z^{-m}\right) :, \tag{3.9}$$

$$S_i^-(z) = e^{Q_i/\beta} z^{s_i^-[0]} : \exp\left(-\sum_{m\neq 0} s_i^-[m] z^{-m}\right) : . \tag{3.10}$$

They satisfy the difference equations:

$$S_i^+(zq^{-r_i}) = t^{-2} q^{2r_i} : A_i(z) S_i^+(zq^{r_i}) :, \tag{3.11}$$

and

$$S_i^-(zt) = t^{-2} q^{2r_i} : A_i(z) S_i^-(zt^{-1}) : . \tag{3.12}$$

Remark on notation. To avoid confusion, let us emphasize that in the case $\mathfrak{g} = A_\ell$ our notation here differs slightly from that of [15]. Namely, our q and t here correspond to $q^{1/2}$ and $(q/p)^{1/2}$, respectively, of [15] (though our β coincides with β of [15]). We made this change of notation to avoid the appearance of half-integers in the formulas. Also, this notation agrees with that of [17].

The connection between our notation and that of [1] is as follows: $x = q/t, r = 1/(1-\beta)$; $q = x^r, t = x^{r-1}$.

3.3. **Definition of $\mathcal{W}_{q,t}(\mathfrak{g})$.** Let $\mathbf{H}_{q,t}(\mathfrak{g})$ be the vector space spanned by formal power series of the form

$$: \partial_z^{n_1} Y_{i_1}(zq^{j_1}t^{k_1})^{\epsilon_1} \dots \partial_z^{n_1} Y_{i_l}(zq^{j_l}t^{k_l})^{\epsilon_l} :, \tag{3.13}$$

where $\epsilon_i = \pm 1$. The pair $(\mathbf{H}_{q,t}(\mathfrak{g}), \pi_0)$ is a *deformed chiral algebra* (DCA) in the sense of [19]. The definition of DCA is recalled in Appendix D.

Denote

$$S_i^+ = \int S_i^+(z) dz : \pi_0 \to \pi_{-\beta\alpha_i/r_i},$$
$$S_i^- = \int S_i^-(z) dz : \pi_0 \to \pi_{\alpha_i}.$$

Here the integral simply means the (-1)st Fourier coefficient of the series $S_i^\pm(z)$, which is a well-defined linear operator. We call these operators the *screening operators.*

We define the DCA $\mathbf{W}_{q,t}(\mathfrak{g})$ as the maximal subalgebra of $(\mathbf{H}_{q,t}(\mathfrak{g}), \pi_0)$, which commutes with the operators $S_i^\pm, i = 1, \dots, \ell$, i.e., the subspace of $\mathbf{H}_{q,t}(\mathfrak{g})$, which consists of all fields that commute with these operators. We define the deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ as the associative algebra, topologically generated by the Fourier coefficients of fields from $\mathbf{W}_{q,t}(\mathfrak{g})$. All elements of the algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ act on the Fock representations π_λ and commute with the screening operators.

We call the field (3.13) *elementary*, if it does not contain derivatives. We assign to such a term the element of the weight lattice of $\mathfrak{g}$,

$$\sum_{a=1}^{\ell} \epsilon_a \omega_{i_a}.$$

Conjecture 1. *Let $\mathfrak{g}$ be a simple Lie algebra and $\widehat{\mathfrak{g}}$ be the corresponding affine Kac-Moody algebra. For each $i = 1, \dots, \ell$, there exists a field $T_i(z)$ in $\mathbf{W}_{q,t}(\mathfrak{g})$, such that $T_i(z) = Y_i(z) +$ the sum of elementary terms of the form*

$$c(q,t) : Y_i(z) A_{i_1}(zq^{a_1}t^{b_1})^{-1} \dots A_{i_k}(zq^{a_k}t^{b_k})^{-1} :$$

(where $c(q,1)$ is a positive integer independent of q). Furthermore, the set of weights of these terms counted with multiplicity $c(q,1)$ is the set of weights of the finite-dimensional irreducible representation V_{ω_i} of $U_q(\widehat{\mathfrak{g}})$ with highest weight ω_i.

Such fields $T_i(z)$ have been constructed in [41, 15, 2] in the A_ℓ case. In Sect. 5 we will explicitly construct the field $T_1(z)$ for all simple Lie algebras of classical types. Some motivations for Conjecture 1 are discussed in Sect. 6.1.

3.4. **Exchange relations.** In this subsection we compute the exchange relations between the fields $T_i(z)$, provided that our Conjecture 1 is true. We assume throughout the rest of this section that $|q| < 1$ and $|t| < 1$.

According to Conjecture 1, each $T_i(z)$ is the sum of $Y_i(z)$ and other terms which are normally ordered products of $Y_i(z)$ and $A_i(zq^at^b)^{-1}$. Let us recall from [19] what we mean by an exchange relation. It follows from the commutation relations (3.1)–(3.3) that the composition $T_i(z)T_j(w)$ converges for $|z| \gg |w|$ and can be analytically continued to a meromorphic function on $\mathbb{C}^\times \times \mathbb{C}^\times$ with poles on shifted diagonals $z = w\gamma$, where $\gamma \in q^{\mathbb{Z}}t^{\mathbb{Z}}$. Let us denote this analytic continuation by $R(T_i(z)T_j(w))$. By exchange relation we understand a relation of the type

$$R(T_i(z)T_j(w)) = S_{T_i,T_j}\left(\frac{w}{z}\right) R(T_j(w)T_i(z)),$$

where $S(z)$ is a meromorphic function.

According to formulas (3.1) and (3.2) the analytic continuations of $Y_i(z)$ and $A_i(w)^{-1}$ satisfy:

$$\begin{aligned} R(Y_i(z)A_j(w)^{-1}) &= R(A_j(w)^{-1}Y_i(z)), \\ R(A_i(z)^{-1}A_j(w)^{-1}) &= R(A_j(w)^{-1}A_i(z)^{-1}), \end{aligned} \tag{3.14}$$

so that fields $A_j(z)^{-1}$ are mutually local and also local with the fields $Y_i(w)$. Formulas (3.14) imply that if $G_i^\alpha(z)$ is one of the terms entering $T_i(z)$ and $G_j^\beta(w)$ is one of the terms entering $T_j(w)$, then

$$S_{G_i^\alpha, G_j^\beta}\left(\frac{w}{z}\right) = S_{Y_i,Y_j}\left(\frac{w}{z}\right),$$

which means that

$$S_{T_i,T_j}\left(\frac{w}{z}\right) = S_{Y_i,Y_j}\left(\frac{w}{z}\right).$$

It is straightforward to find from (3.3) that

$$S_{Y_i,Y_j}\left(\frac{w}{z}\right) = \exp \sum_{n>0} \frac{1}{n}(q^n - q^{-n})(t^n - t^{-n}) M_{ij}(q^n, t^n) \left(\left(\frac{w}{z}\right)^n - \left(\frac{z}{w}\right)^n \right), \tag{3.15}$$

where the matrix $(M_{ij})_{1 \leq i,j \leq \ell}$ is given by formula (3.4). Formula (3.15) is an elliptic function with the multiplicative period $q^{2r^\vee h^\vee} t^{2h}$, where h and $h^\vee$ are the Coxeter and the dual Coxeter numbers, respectively. It can be rewritten as the ratio of products of theta-functions, as in [15] in the case of A_ℓ.

One can also obtain quadratic relations on the Fourier coefficients of various fields from the DCA $\mathbf{W}_{q,t}(\mathfrak{g})$ similar to the relations found in [41, 15, 2]. A general method for writing such relations is described in [19].

3.5. **Relations between the screening currents.** Let $p_i = q^{r_i} t^{-1}$, $r_{ij} = \min(r_i, r_j)$, and $p_{ij} = q^{r_{ij}} t^{-1}$. Denote

$${}^L B_{ij} = \frac{B_{ij}}{r_i r_j}.$$

Introduce the notation

$$\theta(z; a) = \prod_{n=1}^\infty (1 - z q^{n-1})(1 - z^{-1} q^n)(1 - q^n).$$

Direct computation gives us the following relations on the analytic continuations of the compositions of the screening currents $S_i^+(z)$:

$$S_i^+(z) S_i^+(w) = p_i^{-2} \left(\frac{w}{z}\right)^{-{}^L B_{ii}\beta + 2} \frac{\theta\left(\frac{w}{z} p_i^2; q^{2r_{ii}}\right)}{\theta\left(\frac{w}{z} p_i^{-2}; q^{2r_{ii}}\right)} S_i^+(w) S_i^+(z),$$

$$S_i^+(z) S_j^+(w) = -p_{ij} \left(\frac{w}{z}\right)^{-{}^L B_{ij}\beta - 1} \frac{\theta\left(\frac{w}{z} p_{ij}^{-1}; q^{2r_{ij}}\right)}{\theta\left(\frac{w}{z} p_{ij}; q^{2r_{ij}}\right)} S_j^+(w) S_i^+(z), \quad i \neq j, B_{ij} \neq 0,$$

$$S_i^+(z) S_j^+(w) = S_j^+(w) S_i^+(z), \qquad B_{ij} = 0.$$

Let $\widetilde{p}_{ij} = q^{B_{ij}} t^{-1}$. We have the following relations for $S_i^-(z)$:

$$S_i^-(z) S_i^-(w) = p_i^2 \left(\frac{w}{z}\right)^{-B_{ii}/\beta + 2} \frac{\theta\left(\frac{w}{z} p_i^{-2}; t^2\right)}{\theta\left(\frac{w}{z} p_i^2; t^2\right)} S_i^-(w) S_i^-(z),$$

$$S_i^-(z) S_j^-(w) = -\widetilde{p}_{ij}^{-1} \left(\frac{w}{z}\right)^{-B_{ij}/\beta - 1} \frac{\theta\left(\frac{w}{z} \widetilde{p}_{ij}; t^2\right)}{\theta\left(\frac{w}{z} \widetilde{p}_{ij}^{-1}; t^2\right)} S_j^-(w) S_i^-(z), \qquad i \neq j.$$

For $\mathfrak{g} = A_\ell$ these relations were obtained in [15].

One can use the screening currents to construct resolutions of irreducible representations of $\mathcal{W}(\mathfrak{g})$, as in [35, 24, 16] for $\mathfrak{g} = A_\ell$. For that one needs to multiply each screening current by a ratio of theta-functions, to make them single-valued, as in [35, 24, 16].

4. Identification of various limits of $\mathcal{W}_{q,t}(\mathfrak{g})$

4.1. **The conformal limit $q \to 1, t = q^\beta$.** Let $\mathcal{H}_\beta(\mathfrak{g})$ be the Heisenberg algebra with generators $\mathbf{a}_i[n], i = 1, \dots, \ell; n \in \mathbb{Z}$, and relations:

$$[\mathbf{a}_i[n], \mathbf{a}_j[m]] = nB_{ij}\beta\delta_{n,-m}. \tag{4.1}$$

By abuse of notation, we denote by π_μ the Fock representation of $\mathcal{H}_\beta(\mathfrak{g})$ generated by a vector v_μ, such that $\mathbf{a}_i[n]v_\mu = 0, n > 0$, and $\mathbf{a}_i[0]v_\mu = (\mu, \alpha_i)v_\mu$. We also use the same notation as before, e^{Q_i}, for the shift operators acting from π_μ to $\pi_{\mu+\beta\alpha_i}$ and satisfying

$$[\mathbf{a}_i[n], e^{Q_j}] = B_{ij}\beta\delta_{n,0}e^{Q_j}.$$

Now set

$$\mathbf{S}_i^+(z) = e^{-Q_i/r_i} z^{-\mathbf{a}_i[0]/r_i} : \exp\left(\sum_{m\neq 0} \frac{1}{mr_i}\mathbf{a}_i[m]z^{-m}\right) :, \tag{4.2}$$

$$\mathbf{S}_i^-(z) = e^{Q_i/\beta} z^{\mathbf{a}_i[0]/\beta} : \exp\left(-\sum_{m\neq 0} \frac{1}{m\beta}\mathbf{a}_i[m]z^{-m}\right) : . \tag{4.3}$$

Denote

$$\mathbf{S}_i^\pm = \int \mathbf{S}_i^\pm(z)dz.$$

These are the ordinary screening operators.

Let us recall the definition of the ordinary $\mathcal{W}$–algebra associated to $\mathfrak{g}$ from [13, 14]. Let π_0 be the vertex operator algebra (VOA) associated to the Heisenberg algebra $\mathcal{H}_\beta(\mathfrak{g})$ (see [14]). For generic β the VOA $\mathbf{W}_\beta(\mathfrak{g})$ is defined as the vertex operator subalgebra of the VOA π_0, which is the intersection of kernels of the screening operators $\mathbf{S}_i^-, i = 1, \dots \ell$:

$$\mathbf{W}_\beta(\mathfrak{g}) = \bigcap_{i=1}^{\ell} \operatorname{Ker} \mathbf{S}_i^-.$$

Thus, $\mathbf{W}_\beta(\mathfrak{g})$ consists of the fields that commute with $\mathbf{S}_i^-$. The $\mathcal{W}$–algebra $\mathcal{W}_\beta(\mathfrak{g})$ is defined as the associative (or Lie) algebra generated by the Fourier coefficients of the fields from $\mathbf{W}_\beta(\mathfrak{g})$.

Remark 1. In the notation of [14], our $\mathcal{W}_\beta(\mathfrak{g})$ is $\mathcal{W}_\gamma(\mathfrak{g})$, where $\gamma = (r^\vee/\beta)^{1/2}$. □

It was proved in [13] (see also [14]) that for generic β, $\mathbf{W}_\beta(\mathfrak{g})$ automatically lies in the kernel of the other set of screening operators, $\mathbf{S}_i^+, i = 1, \dots, \ell$. This implies the following duality.

Theorem 1 ([13]). *For generic β,*

$$\mathcal{W}_\beta(\mathfrak{g}) \simeq \mathcal{W}_{r^\vee/\beta}({}^L\mathfrak{g}),$$

where ${}^L\mathfrak{g}$ is the Langlands dual Lie algebra to $\mathfrak{g}$.

Now let us consider the limit of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $q \to 1$ with $t = q^\beta$. Let us pass to a new set of generators

$$\widetilde{a}_i[n] = \frac{a_i[n]}{q - q^{-1}}.$$

It is clear from formula (3.1) that in the limit the operators $\widetilde{a}_i[n]$ satisfy the relations given by formula (4.1). Hence we can identify $\widetilde{a}_i[n]$ with $\mathbf{a}_i[n]$ in this limit. Furthermore, it is easy to see that the limit of the field $S_i^\pm(z)$ equals $\mathbf{S}_i^\pm(z)$. Comparing the definitions of $\mathcal{W}_{q,t}(\mathfrak{g})$ and $\mathcal{W}_\beta(\mathfrak{g})$, we obtain

Theorem 2. *In the limit $q \to 1$ with $t = q^\beta$, $\mathcal{W}_{q,t}(\mathfrak{g})$ becomes $\mathcal{W}_\beta(\mathfrak{g})$.*

It is clear from the definition that $\mathcal{W}_{q,t}(\mathfrak{g}) \simeq \mathcal{W}_{q^{-1},t^{-1}}(\mathfrak{g})$ for all $\mathfrak{g}$, and for simply-laced $\mathfrak{g}$,

$$\mathcal{W}_{q,t}(\mathfrak{g}) \simeq \mathcal{W}_{t,q}(\mathfrak{g}).$$

The latter is the analogue of Theorem 1 (note that ${}^L\mathfrak{g} = \mathfrak{g}$ in the simply-laced case). However, for nonsimply-laced $\mathfrak{g}$, $\mathcal{W}_{q,t}(\mathfrak{g})$ is *not* isomorphic to $\mathcal{W}_{t,q}({}^L\mathfrak{g})$.

4.2. **The first classical limit $t \to 1$.** Let us consider the limit $\mathcal{W}_{q,1}(\mathfrak{g})$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $t \to 1$ (q is fixed). Both $\mathcal{H}_{q,1}(\mathfrak{g})$ and $\mathcal{W}_{q,1}(\mathfrak{g})$ are commutative algebras, but they inherit a Poisson structure. The Poisson-Heisenberg algebra $\mathcal{H}_{q,1}(\mathfrak{g})$ has the relations

$$\{a_i[n], a_j[n]\} = (q^{B_{ij}n} - q^{-B_{ij}n})\delta_{n,-m}.$$

The Poisson algebra $\mathcal{H}_{q,1}$ was introduced in [17]. $\mathcal{W}_{q,1}(\mathfrak{g})$ is its Poisson subalgebra. It is easy to see that the limit $q \to 1$ of $\mathcal{W}_{q,1}(\mathfrak{g})$ is isomorphic to the limit of $\mathcal{W}_\beta(\mathfrak{g})$ as $\beta \to 0$. The latter is known to be isomorphic to the center of $U(\widehat{\mathfrak{g}})$ at the critical level and to $\mathcal{W}({}^L\mathfrak{g})$ [13].

Conjecture 2. *The limit $t \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ with fixed q is isomorphic, as a Poisson algebra, to the center of $U_q(\widehat{\mathfrak{g}})$ at the critical level.*

Conjecture 2 was proved in [17] for $\mathfrak{g} = A_\ell$. It has been verified in [18] for $\mathfrak{g} = B_\ell, C_\ell$ and in [28] for $\mathfrak{g} = D_\ell, E_6$ and G_2. Some of these results are given in Appendix A.

The following observation also confirms the conjecture. Consider the quantized enveloping algebra $U_q(\widehat{\mathfrak{g}})$ and its loop-like generators (Drinfeld's new generators), $\kappa_{i,n}$ in the notation of [11]. In the limit when the level tends to 0, they generate a Poisson algebra $\mathcal{B}_q(\mathfrak{g})$. In [17] we explained that the free field realization of $U_q(\widehat{\mathfrak{g}})$ induces an embedding of the center of $U_q(\widehat{\mathfrak{g}})$ into a Heisenberg-Poisson algebra $\mathcal{A}_q(\mathfrak{g})$, which is part of the free field realization of $U_q(\widehat{\mathfrak{g}})$. In the case $\mathfrak{g} = A_\ell$, when the free field realization is available [3], we know that this Heisenberg-Poisson algebra $\mathcal{A}_q(\mathfrak{g})$ is isomorphic to $\mathcal{B}_q(\mathfrak{g})$. We expect that the same is true for other $\mathfrak{g}$. But it is easy to see that $\mathcal{B}_q(\mathfrak{g})$ is isomorphic to $\mathcal{H}_{q,1}(\mathfrak{g})$.

4.3. **The second classical limit $q \to 1$.** Now consider the limit $\mathcal{W}_{1,t}(\mathfrak{g})$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ as $q \to 1$ (t is fixed). This is again a Poisson algebra, which is a Poisson subalgebra of $\mathcal{H}_{1,t}(\mathfrak{g})$. The latter has the following relations:

$$\{a_i[n], a_j[n]\} = r_i((t + t^{-1})\delta_{ij} - I_{ij})\delta_{n,-m}.$$

The $t \to 1$ limit of $\mathcal{W}_{1,t}(\mathfrak{g})$ coincides with the $\beta \to \infty$ limit of $\mathcal{W}_\beta(\mathfrak{g})$, which is isomorphic to the classical $\mathcal{W}$–algebra $\mathcal{W}(\mathfrak{g})$ obtained by the Drinfeld-Sokolov reduction of the dual space to $\widehat{\mathfrak{g}}$ (see [13]). On the other hand, in [21, 40] a p–difference analogue of the Drinfeld-Sokolov reduction was defined as a Poisson reduction of the Poisson-Lie group $\mathcal{G} = G((z))$. The result of this reduction is a p–deformation of $\mathcal{W}(\mathfrak{g})$, which we denote by $\mathcal{W}^p(\mathfrak{g})$. There is also an embedding of $\mathcal{W}^p(\mathfrak{g})$ into a Heisenberg-Poisson algebra; this is a difference analogue of the Miura transformation from [12]. These constructions are recalled in Appendix B.

Conjecture 3. *The limit $q \to 1$ of $\mathcal{W}_{q,t}(\mathfrak{g})$ with fixed t is isomorphic to $\mathcal{W}^p(\mathfrak{g})$, where $p = t^{r^\vee}$. Furthermore, the free field realization of $\mathcal{W}_{1,t}(\mathfrak{g})$ coincides with the difference Miura transformation of $\mathcal{W}^p(\mathfrak{g})$.*

For simply-laced $\mathfrak{g}$, the two Poisson algebras obtained as limits $q \to 1$ and $t \to 1$ of $W_{q,t}(\mathfrak{g})$ are isomorphic: $\mathcal{W}_{q,1}(\mathfrak{g}) \simeq \mathcal{W}_{1,q}(\mathfrak{g})$. On the other hand, for $\mathfrak{g} = A_\ell$, it follows from [21, 40] that $\mathcal{W}_{t,1}(A_\ell)$ is the Poisson algebra obtained by the t–difference Drinfeld-Sokolov reduction of $\widehat{SL}_{\ell+1}$. Hence Conjecture 3 holds for $\mathfrak{g} = A_\ell$.

In Appendix B we check Conjecture 3 for $\mathfrak{g} = C_2$.

Remark 2. We know that the center of $U(\widehat{\mathfrak{g}})$ at the critical level is isomorphic to $\mathcal{W}({}^L\mathfrak{g})$. However, for nonsimply-laced $\mathfrak{g}$ and $q \neq 1$, the center of $U_q(\widehat{\mathfrak{g}})$ is not isomorphic to $\mathcal{W}_{1,q}({}^L\mathfrak{g})$. $\square$

4.4. **The limit** $q \to \epsilon$**.** Let $\epsilon = 1$ for simply-laced $\mathfrak{g}$, and $\epsilon = \exp(\pi i/r^\vee)$ for nonsimply-laced $\mathfrak{g}$. Since we have already discussed the limit $q \to 1$, we can now focus on nonsimply-laced $\mathfrak{g}$. Let R_s (respectively, R_l) be the subset of $\{1, 2, \dots, \ell\}$, consisting of the labels of short (respectively, long) simple roots. Note that $r_i = 2$ for $i \in R_s$, and $r_i = 2r^\vee$ for $i \in R_l$. The inspection of formulas (3.1), (3.2), and (3.3) shows that although the algebra $\mathcal{H}_{q,t}(\mathfrak{g})$ is not commutative when $q = \epsilon$, it contains a large center $\mathcal{H}'_t(\mathfrak{g})$, generated by $a_i[n], i \in R_l, n \in \mathbb{Z}$, and $a_i[r^\vee n], i \in R_s, n \in \mathbb{Z}$. The center $\mathcal{H}'_t(\mathfrak{g})$ has a natural Poisson structure, with the Poisson brackets between the generators given by

$$\begin{aligned}
\{a_i[n], a_i[m]\} &= (-1)^{n+1}(t^{2n} - t^{-2n})\delta_{n,-m}, && i \in R_l,\\
\{a_i[r^\vee n], a_i[r^\vee m]\} &= (-1)^{n+1}\frac{1}{r^\vee}(t^{2r^\vee n} - t^{-2r^\vee n})\delta_{n,-m}, && i \in R_s,\\
\{a_i[n], a_j[m]\} &= (-1)^{n}(t^{n} - t^{-n})\delta_{n,-m}, && i,j \in R_l; I_{ij} \neq 0,\\
\{a_i[r^\vee n], a_j[m]\} &= (-1)^{n}(t^{r^\vee n} - t^{-r^\vee n})\delta_{r^\vee n,-m}, && i \in R_s, j \in R_l; I_{ij} \neq 0,\\
\{a_i[r^\vee n], a_j[r^\vee m]\} &= (-1)^{n}\frac{1}{r^\vee}(t^{r^\vee n} - t^{-r^\vee n})\delta_{n,-m}, && i,j \in R_s; I_{ij} \neq 0.
\end{aligned}$$

Now the Dynkin diagram of $\mathfrak{g}$ is obtained by folding from the Dynkin diagram of a simply-laced Lie algebra $\widetilde{\mathfrak{g}}$ under the action of an automorphism of order $r^\vee$. Let Γ be the set of vertices of the Dynkin diagram of $\widetilde{\mathfrak{g}}$, and $\sigma : \Gamma \to \Gamma$ be the corresponding automorphism of order $r^\vee$. Denote by $(\widetilde{B}_{ij})$ the Cartan matrix of $\widetilde{\mathfrak{g}}$. Consider the

Poisson algebra $\widetilde{\mathcal{H}}_t(\mathfrak{g})$ with generators $b_i[n], i \in \Gamma, n \in \mathbb{Z}$, and relations:

$$\{b_i[n], b_j[m]\} = \sum_{k=0}^{r^\vee} (t^{n\widetilde{B}_{i,\sigma(j)}} - t^{-n\widetilde{B}_{i,\sigma(j)}})\delta_{n,-m}, \tag{4.4}$$

$$b_i[n] = \epsilon^n b_{\sigma(i)}[n]. \tag{4.5}$$

We claim that the Poisson algebras $\mathcal{H}'_t(\mathfrak{g})$ and $\widetilde{\mathcal{H}}_t(\mathfrak{g})$ are isomorphic. Indeed, for each $i \in \{1, 2, \dots, \ell\}$, choose an element $\overline{i}$ of the orbit of σ in Γ corresponding to the simple root α_i. The long simple roots of $\mathfrak{g}$ correspond to simply-transitive orbits of σ in Γ, while short simple roots correspond to the fixed points of σ in Γ. Due to formula (4.5), we can choose $b_{\overline{i}}[n], i \in R_l; n \in \mathbb{Z}$, and $\frac{1}{r^\vee} b_{\overline{i}}[r^\vee n], i \in R_s; n \in \mathbb{Z}$, as the linearly independent generators of $\widetilde{\mathcal{H}}_t(\mathfrak{g})$. It is easy to see that the Poisson brackets between them coincide with the Poisson brackets between the generators of $\mathcal{H}'_t(\mathfrak{g})$ up to a sign, which can be removed by an elementary redefinition of the generators.

Now let ${}^L\widehat{\mathfrak{g}}$ be the twisted affine algebra, which is Langlands dual to $\widehat{\mathfrak{g}}$, i.e., the Cartan matrix of ${}^L\widehat{\mathfrak{g}}$ is dual to the Cartan matrix of $\widehat{\mathfrak{g}}$. Consider the quantized enveloping algebra $U_t({}^L\widehat{\mathfrak{g}})$ and its Drinfeld's generators $\kappa_{i,n}$ [11] (see Sect. 4.2). In the limit when the level of $U_t({}^L\widehat{\mathfrak{g}})$ tends to 0, they generate a Poisson algebra $\mathcal{B}_t({}^L\widehat{\mathfrak{g}})$. It is easy to see that $\mathcal{B}_t({}^L\widehat{\mathfrak{g}})$ is isomorphic to $\widetilde{\mathcal{H}}_t(\mathfrak{g})$, and hence to $\mathcal{H}'_t(\mathfrak{g})$. In view of the last paragraph of Sect. 4.2, it is natural to make the following conjecture.

Let $\mathcal{W}'_t(\mathfrak{g})$ be the intersection of $\mathcal{W}_{\epsilon,t}(\mathfrak{g})$ and $\mathcal{H}'_t(\mathfrak{g})$. This is a Poisson subalgebra of $\mathcal{H}'_t(\mathfrak{g})$.

Conjecture 4. $\mathcal{W}'_t(\mathfrak{g})$ *is isomorphic, as a Poisson algebra, to the center of* $U_t({}^L\widehat{\mathfrak{g}})$ *at the critical level.*

Some evidence supporting this conjecture will be presented in Sect. 6.2.

5. Deformed $\mathcal{W}$–algebras associated to Lie algebras of classical types

5.1. **The fields $\Lambda_i(z)$.** We first define a set J and fields $\Lambda_i(z), i \in J$, for each simple Lie algebra of classical type.

5.1.1. *The A_ℓ series.* $J = \{1, \dots, \ell+1\}$.

$$\Lambda_i(z) =: Y_i(zq^{-i+1}t^{i-1}) Y_{i-1}(zq^{-i}t^i)^{-1} :, \qquad i = 1, \dots, \ell+1.$$

Equivalently,

$$\Lambda_1(z) = Y_1(z),$$
$$\Lambda_i(z) =: \Lambda_i(z) A_{i-1}(zq^{-i+1}t^{i-1})^{-1} :, \qquad i = 2, \dots, \ell.$$

5.1.2. *The B_ℓ series.* $J = \{1, \dots, \ell, 0, \overline{\ell}, \dots, \overline{1}\}$.

$$
\begin{aligned}
\Lambda_i(z) &=: Y_i(zq^{-2i+2}t^{i-1})Y_{i-1}(zq^{-2i}t^i)^{-1} :, \qquad i=1,\dots,\ell-1,\\
\Lambda_\ell(z) &=: Y_\ell(zq^{-2\ell+3}t^{\ell-1})Y_\ell(zq^{-2\ell+1}t^{\ell-1})Y_{\ell-1}(zq^{-2\ell}t^\ell)^{-1} :,\\
\Lambda_0(z) &= \frac{(q+q^{-1})(qt^{-1}-q^{-1}t)}{q^2t^{-1}-q^{-2}t} : Y_\ell(zq^{-2\ell+3}t^{\ell-1})Y_\ell(zq^{-2\ell-1}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{\ell}}(z) &=: Y_{\ell-1}(zq^{-2\ell+2}t^\ell)Y_\ell(zq^{-2\ell+1}t^{\ell+1})^{-1}Y_\ell(zq^{-2\ell-1}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{i}}(z) &=: Y_{i-1}(zq^{-4\ell+2i+2}t^{2\ell-i})Y_i(zq^{-4\ell+2i}t^{2\ell-i+1})^{-1} :, \qquad i=1,\dots,\ell-1.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\Lambda_1(z) &= Y_1(z),\\
\Lambda_i(z) &=: \Lambda_{i-1}(z)A_{i-1}(zq^{-2i+2}t^{i-1})^{-1} :, \qquad i=2,\dots,\ell,\\
\Lambda_0(z) &= \frac{(q+q^{-1})(qt^{-1}-q^{-1}t)}{q^2t^{-1}-q^{-2}t} : \Lambda_\ell(z)A_\ell(zq^{-2\ell}t^\ell)^{-1} :,\\
\Lambda_{\overline{\ell}}(z) &= \frac{q^2t^{-1}-q^{-2}t}{(q+q^{-1})(qt^{-1}-q^{-1}t)} : \Lambda_0(z)A_\ell(zq^{-2\ell+2}t^\ell)^{-1} :,\\
\Lambda_{\overline{i}}(z) &=: \Lambda_{\overline{i+1}}(z)A_i(zq^{-2(2\ell-i-1)}t^{2\ell-i})^{-1}, \qquad i=1,\dots,\ell-1.
\end{aligned}
$$

5.1.3. *The C_ℓ series.* $J=\{1,\dots,\ell,\overline{\ell},\dots,\overline{1}\}$.

$$
\begin{aligned}
\Lambda_i(z) &=: Y_i(zq^{-i+1}t^{i-1})Y_{i-1}(zq^{-i}t^i)^{-1} :, \qquad i=1,\dots,\ell,\\
\Lambda_{\overline{i}}(z) &=: Y_{i-1}(zq^{-2\ell+i-2}t^{2\ell-i})Y_i(zq^{-2\ell+i-3}t^{2\ell-i+1})^{-1} :, \qquad i=1,\dots,\ell.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\Lambda_1(z) &= Y_1(z),\\
\Lambda_i(z) &=: \Lambda_{i-1}(z)A_{i-1}(zq^{-i+1}t^{i-1})^{-1} :, \qquad i=2,\dots,\ell,\\
\Lambda_{\overline{\ell}}(z) &=: \Lambda_\ell(z)A_\ell(zq^{-\ell-1}t^\ell)^{-1} :,\\
\Lambda_{\overline{i}}(z) &=: \Lambda_{\overline{i+1}}(z)A_i(zq^{-2\ell+i-2}t^{2\ell-i})^{-1} :, \qquad i=1,\dots,\ell-1.
\end{aligned}
$$

5.1.4. *The D_ℓ series.* $J=\{1,\dots,\ell,\overline{\ell},\dots,\overline{1}\}$.

$$
\begin{aligned}
\Lambda_i(z) &=: Y_i(zq^{-i+1}t^{i-1})Y_{i-1}(zq^{-i}t^i)^{-1} :, \qquad i=1,\dots,\ell-2,\\
\Lambda_{\ell-1}(z) &=: Y_\ell(zq^{-\ell+2}t^{\ell-2})Y_{\ell-1}(zq^{-\ell+2}t^{\ell-2})Y_{\ell-2}(zq^{-\ell+1}t^{\ell-1})^{-1} :,\\
\Lambda_\ell(z) &=: Y_\ell(zq^{-\ell+2}t^{\ell-2})Y_{\ell-1}(zq^{-\ell}t^\ell)^{-1} :,\\
\Lambda_{\overline{\ell}}(z) &=: Y_{\ell-1}(zq^{-\ell+2}t^{\ell-2})Y_\ell(zq^{-\ell}t^\ell)^{-1} :,\\
\Lambda_{\overline{\ell-1}}(z) &=: Y_{\ell-2}(zq^{-\ell+1}t^{\ell-1})Y_{\ell-1}(zq^{-\ell}t^\ell)^{-1}Y_\ell(zq^{-\ell}t^\ell) :,\\
\Lambda_{\overline{i}}(z) &=: Y_{i-1}(zq^{-2\ell+i+2}t^{2\ell-i-2})Y_i(zq^{-2\ell+i+1}t^{2\ell-i-1})^{-1} :, \qquad i=1,\dots,\ell-2.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\Lambda_1(z) &= Y_1(z),\\
\Lambda_i(z) &=: \Lambda_{i-1}(z) A_{i-1}(zq^{-i+1}t^{i-1})^{-1} :, \qquad i=2,\dots,\ell,\\
\Lambda_{\bar{\ell}}(z) &=: \Lambda_{\ell-1}(z) A_{\ell}(zq^{-\ell+1}t^{\ell-1})^{-1} :,\\
\Lambda_{\overline{\ell-1}}(z) &=: \Lambda_{\bar{\ell}}(z) A_{\ell-1}(zq^{-\ell+1}t^{\ell-1})^{-1} :\\
&=: \Lambda_{\ell}(z) A_{\ell}(zq^{-\ell+1}t^{\ell-1})^{-1} :,\\
\Lambda_{\bar{i}}(z) &=: \Lambda_{\overline{i+1}}(z) A_i(zq^{-2\ell+i+2}t^{2\ell-i-2})^{-1} :, \qquad i=1,\dots,\ell-2.
\end{aligned}
$$

5.2. **The first generating field of the deformed $\mathcal{W}$–algebra.** For each Lie algebra of classical type, set

$$T_1(z) = \sum_{i\in J} \Lambda_i(z). \tag{5.1}$$

In the A_ℓ case the field $T_1(z)$ coincides with the one obtained in [41, 15, 2].

Theorem 3. *For each Lie algebra $\mathfrak{g}$ of classical type, the field $T_1(z)$ commutes with the screening operators $S_i^\pm, i=1,\dots,\ell$ and hence belongs to $\mathbf{W}_{q,t}(\mathfrak{g})$.*

Proof. As in [15, 2], we need to prove that the commutator of $T_1(z)$ and $S_i^\pm(w)$, considered as a formal power series in z and w, is a total difference with respect to w:

$$
\begin{aligned}
[T_1(z), S_i^+(w)] &= \mathcal{D}_{q^2}\cdot R_i^+(z,w),\\
[T_1(z), S_i^-(w)] &= \mathcal{D}_{t^2}\cdot R_i^-(z,w),
\end{aligned}
$$

where

$$\mathcal{D}_a \cdot f(w) = \frac{f(w)-f(wa)}{w(1-a)}.$$

This will immediately imply the statement of the theorem.

This fact has already been proved in [15, 2] in the case $\mathfrak{g}=A_\ell$ not only for $T_1(z)$, but for all $T_i(z), i=1,\dots,\ell$. Hence we focus on the remaining series B_ℓ, C_ℓ, and D_ℓ.

Note that according to formula (3.2),

$$[y_i[n], S_j^\pm(w)] = 0, \qquad i\neq j.$$

It is then clear from the explicit formulas for $T_1(z)$ that nontrivial contribution to the commutator $[T_1(z), S_i^+(w)]$, where $i=1,\dots,\ell-1$ (resp., $i=1,\dots,\ell-2$) in the case $\mathfrak{g}=B_\ell, C_\ell$ (resp, $\mathfrak{g}=D_\ell$) come from the terms $\Lambda_i(z), \Lambda_{i+1}(z)$, and $\Lambda_{\bar{i}}(z), \Lambda_{\overline{i+1}}(z)$. It is easy to see from the formula for $T_1(z)$ that the corresponding contributions coincide with the commutators of $S_i^\pm(w)$ with $T_1(z)$ and $T_{\ell-1}(z)$, respectively, in the case $\mathfrak{g}=A_\ell$. Hence they are total differences according to [15, 2].

For the remaining screening currents: $S_\ell^\pm(z)$ in the case $\mathfrak{g}=B_\ell, C_\ell$, and $S_{\ell-1}^\pm(z)$, $S_\ell^\pm(z)$ in the case $\mathfrak{g}=D_\ell$, it suffices to prove the statement when $\mathfrak{g}$ is of types B_2, C_2, and $\mathfrak{g}=D_3$. In the latter case, the field $T_1(z)$ is the same as the field $T_2(z)$ for $\mathfrak{g}=A_3$, and hence this case follows again from [15, 2]. Thus, we are left with $\mathfrak{g}=B_2, C_2$ and $S_2^\pm(w)$. Here we explain the proof in the most difficult case $\mathfrak{g}=B_2$ and $S_2^-(w)$. In the remaining cases the statement is proved by a similar computation.

We find using formula (3.2):

$$\Lambda_2(z)S_2^-(w) = q^4 \frac{1-\frac{w}{z}t^{-1}}{1-\frac{w}{z}q^4t^{-1}} : \Lambda_2(z)S_2^-(w) :, \qquad |z| \gg |w|,$$
$$S_2^-(w)\Lambda_2(z) = \frac{1-\frac{z}{w}t}{1-\frac{z}{w}q^{-4}t} : \Lambda_2(z)S_2^-(w) :, \qquad |w| \gg |z|.$$

This implies the following commutator between the formal power series $\Lambda_2(z)$ and $S_2^-(w)$:

$$[\Lambda_2(z), S_2^-(w)] = (q^4-1)\delta\left(\frac{w}{z}q^4t^{-1}\right) : Y_2(wq^3)Y_2(wq)Y_1(wt)^{-1}S_2^-(w) : . \tag{5.2}$$

We obtain in an analogous way:

$$[\Lambda_0(z), S_2^-(w)] = (q^{-2}-q^2)\delta\left(\frac{w}{z}q^4t^{-3}\right) : Y_2(wq^3t^{-2})Y_2(wq^{-1})^{-1}S_2^-(w) : \tag{5.3}$$
$$+ (q^2-q^{-2})\delta\left(\frac{w}{z}q^2t^{-1}\right) : Y_2(wq)Y_2(wq^{-3}t^2)^{-1}S_2^-(w) :, \tag{5.4}$$

and

$$[\Lambda_{\overline{2}}(z), S_2^-(w)] = (q^{-4}-1)\delta\left(\frac{w}{z}q^2t^{-3}\right) : Y_1(wt^{-1})Y_2(wq^{-1})^{-1}Y_2(wq^{-3})^{-1}S_2^-(w) : . \tag{5.5}$$

Now recall that

$$S_2^-(zt) = t^{-2}q^2 : A_2(z)S_2^-(zt^{-1}) :$$

and note that

$$A_2(z) =: Y_1(z)^{-1}Y_2(zqt^{-1})Y_2(zq^{-1}t) :$$

(see formula (3.5)). Using these formulas, we find that the terms (5.2) and (5.3) combine into a total t^2–difference, and so do the terms (5.4) and (5.5). Hence the result is proved. $\square$

Theorem 3 proves Conjecture 1 for all $\mathfrak{g}$ of classical types and their first fundamental representation.

6. Connection with analytic Bethe Ansatz

Now we discuss the connection between our formulas for $T_1(z)$ and the analytic Bethe Ansatz. We expect the same connection to hold not only for $\mathfrak{g}$ of classical types, but for all $\mathfrak{g}$.

6.1. **The limit $t \to 1$.** The specialization of our formula for $T_1(z)$ to $t=1$ coincides with the Bethe Ansatz formula for the eigenvalues of the transfer-matrix corresponding to the finite-dimensional representation V_{ω_1} of $U_q(\widehat{\mathfrak{g}})$ with highest weight ω_1. Let us explain that in more detail.

For a finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$, the Bethe Ansatz formula for the corresponding eigenvalue $t_V(z)$ is given by a linear combination of terms of the form

$$Y_{i_1}(zq^{a_1})\dots Y_{i_k}(zq^{a_k}),$$

where $Y_i(z)$ are certain functions, such that the set of weights of these terms is the set of weights of V (see [37, 38, 29] for details). If we set $q = 1$, then $t_V(z)$ becomes simply the character of V considered as a representation of $U_q(\mathfrak{g})$. Therefore $t_V(z)$ can be viewed as a "q–character" of the representation V. These q–characters satisfy the following natural properties: $t_{V\oplus W}(z) = t_V(z) + t_W(z)$, $t_{V\otimes W}(z) = t_V(z)t_W(z)$, and $t_{V(\lambda)} = t_V(z\lambda)$, where $V(\lambda)$ is the standard twist of V by $\lambda \in \mathbb{C}^\times$.

Systematically, these q–characters can be obtained as follows. Each finite-dimensional representation V of $U_q(\widehat{\mathfrak{g}})$ gives rise to a generating series $T_V(z)$ of central elements of $U_q(\widehat{\mathfrak{g}})$ at the critical level [39] (see also [10]). In [17] we showed that the free field realization of $U_q(A_\ell^{(1)})$ gives us an embedding of the center of $U_q(A_\ell^{(1)})$ into the Heisenberg-Poisson algebra $\mathcal{H}_{q,1}(A_\ell)$, and that the formula for the image of $T_{V_{\omega_i}}(z)$ in $\mathcal{H}_{q,1}(\mathfrak{g})$ coincides with the formula for $t_{V_{\omega_i}}(z)$ obtained by the Bethe Ansatz. Although the free field realization of $U_q(\widehat{\mathfrak{g}})$ has not yet been constructed for $\mathfrak{g}$ other than A_ℓ, we expect that it exists for arbitrary $\mathfrak{g}$. Furthermore, we expect that one can reproduce the Bethe Ansatz formulas by applying the free field realization to the generating series $T_V(z)$ of elements of the center of $U_q(\widehat{\mathfrak{g}})$. The fact that the limit $t \to 1$ of our formulas for $T_1(z)$ agrees with the Bethe Ansatz formula $t_{V_{\omega_1}}(z)$ therefore provides supporting evidence for Conjecture 2 and a motivation for Conjecture 1.

Our derivation of the formula for $T_1(z)$ suggests two features of the Bethe Ansatz formulas that appear to be new. First, we represent the terms $\Lambda_i(z)$ in the formula for the eigenvalues as the products of $Y_1(z)$ corresponding to the highest weight, and the "step operators" $A_i(z)^{-1}$ corresponding to the simple roots. Second, we show that the linear combination $T_1(z)$ of the terms $\Lambda_i(z)$ is distinguished by the fact that it commutes with the screening operators S_i^+, which are well-defined in the limit $t \to 1$.

We expect that Conjecture 1 has the following generalization: to each integral dominant weight λ of $\mathfrak{g}$ one can attach a field $T_\lambda(z)$ from $\mathbf{W}_{q,t}(\mathfrak{g})$, which is the sum of terms corresponding to weights in the irreducible representation V_λ counted with multiplicity. The limit of $T_\lambda(z)$ as $t \to 1$ should coincide with $t_{V_\lambda}(z)$, and hence one obtains a new method of singling out the "q–characters" by their property of commutativity with the screening operators S_i^+. We will discuss this in more detail in [20].

In the case of A_ℓ, explicit formulas for other generating fields $T_i(z), i = 2, \ldots \ell$, have been found in [17, 15, 2]. Their specialization at $t = 1$ coincides with q–characters of the finite-dimensional representations of A_ℓ with highest weights $\omega_i, i = 2, \ldots \ell$. In fact, these fields can be obtained from the field $T_1(z)$ by a "fusion procedure", i.e., by taking the residues in the operator product expansions of the fields $T_i(z)$. Moreover, for $\mathfrak{g} = A_\ell$ it seems that all fields $T_\lambda(z)$ can be constructed starting from $T_1(z)$ via the fusion procedure (some examples have been given in [19]), and so $\mathbf{W}_{q,t}(A_\ell)$ is the smallest subalgebra of the DCA $(\mathbf{H}_{q,t}(\mathfrak{g}), \pi_0)$, containing $T_1(z)$. For general $\mathfrak{g}$, it would be interesting to analyze which fields $T_\lambda(z)$ can be obtained from $T_1(z)$ by the fusion procedure.

We also remark that our formulas above and formulas from [15, 2] suggest that the fields $T_\lambda(z)$ should have the following symmetry: if we replace $q \to q^{-1}, t \to t^{-1}$, and $Y_i(z) \to Y_i(z)^{-1}$, then $T_\lambda(z)$ becomes $T_{\overline{\lambda}}(z)$, where $\overline{\lambda}$ is the highest weight of the

representation dual to V_λ, up to an overall multiplication of z by a factor $q^a t^b$. In our formulas above, this factor equals $q^{r^\vee h^\vee} t^{-h}$.

6.2. **The limit** $q \to \epsilon$**.** For simply-laced $\mathfrak{g}$, we have the duality $\mathcal{W}_{q,t}(\mathfrak{g}) \simeq \mathcal{W}_{t,q}(\mathfrak{g})$. Hence $\mathcal{W}_{1,t}(\mathfrak{g})$ is isomorphic to $\mathcal{W}_{t,1}(\mathfrak{g})$. Thus, without loss of generality in this subsection we can focus on nonsimply-laced $\mathfrak{g}$.

Recall that according to Conjecture 4, the subalgebra $\mathcal{W}'_t(\mathfrak{g})$ of $\mathcal{W}_{\epsilon,t}(\mathfrak{g})$ is isomorphic to the center of $U_t({}^L\widehat{\mathfrak{g}})$. Just as in the non-twisted case (see the previous subsection), to every finite-dimensional representation V of $U_t({}^L\widehat{\mathfrak{g}})$ corresponds a Bethe Ansatz formula – its "t–character". On the other hand, it also gives rise to a generating series $T_V(z)$ of central elements of $U_t({}^L\widehat{\mathfrak{g}})$. We expect that $U_t({}^L\widehat{\mathfrak{g}})$ has a free field realization, which embeds the center of $U_t({}^L\widehat{\mathfrak{g}})$ into a Heisenberg-Poisson algebra, so that the image of $T_V(z)$ coincides with the formula for the t–character of V.

Therefore we expect that the formulas for the elements of $\mathcal{W}_{q,t}(\mathfrak{g})$ that lie in $\mathcal{W}'_t(\mathfrak{g})$ in the limit $q \to 1$, coincide with the corresponding Bethe Ansatz formulas. Examples of the latter formulas are known in the literature [38, 30], and we can compare them with our formulas.

Let us first consider the formula for $T_1(z)$ in the case of B_ℓ. In this case, all simple roots, except for the ℓth root, are long and the ℓth root is short. Hence an element of $\mathcal{W}_{q,t}(B_\ell)$ lands in $\mathcal{W}'_t(B_\ell)$ as $q \to i$ if $Y_\ell(z)$ appears in it only through combination $Y_\ell(z)Y_\ell(zq^2)$, giving $Y_\ell(z)Y_\ell(-z)$ after the specialization $q = i$. We see from our formula that this is so for $T_1(z)$. Furthermore, the term $\Lambda_0(z)$ vanishes at $q = i$, so we are left with 2ℓ terms. Comparison with [38, 30] shows perfect agreement between the resulting formula for $T_1(z)$ and the Bethe Ansatz formula for the 2ℓ–dimensional representation of $U_t(A^{(2)}_{2\ell-1})$ (note that ${}^L(B^{(1)}_\ell) = A^{(2)}_{2\ell-1}$). This provides supporting evidence for Conjecture 4.

Thus, $T_1(z)$ "interpolates" between the q–character of the $(2\ell+1)$–dimensional representation of $U_q(B^{(1)}_\ell)$ (in the limit $t \to 1$) and the t–character of the 2ℓ–dimensional representation of $U_t(A^{(2)}_{2\ell-1})$.

In the case of C_ℓ, the short simple roots are $\alpha_1, \dots, \alpha_{\ell-1}$, so for an element of $\mathcal{W}_{q,t}(C_\ell)$ to land in $\mathcal{W}'_t(C_\ell)$ as $q \to i$, the fields $Y_i(z)$ should appear in it only in combination $Y_i(z)Y_i(zq^2)$ for $i = 1, \dots, \ell-1$. Inspection of our formula for $T_1(z)$ shows that it does not satisfy this condition. However, the specializations to $q = i$ of other fields from $\mathbf{W}_{q,t}(C_\ell)$ may well satisfy it, in which case they should coincide with the t–characters of certain representations of $U_t(D^{(2)}_{\ell+1})$, since ${}^L(C^{(1)}_\ell) = D^{(2)}_{\ell+1}$.

We have constructed such a field in the case $\mathfrak{g} = C_2$. In the notation of the previous subsection, this is the field $T_{2\omega_1}(z)$. When $t = 1$ it coincides with the q–character of the representation $V_{2\omega_1}$ of $U_q(C^{(1)}_2)$, while at $q = i$ it coincides with the t–character of the representation V_{ω_1} of $U_t(D^{(2)}_3)$. We expect that the same is true when $\mathfrak{g} = C_\ell$.

It would be interesting to see how this duality works for general representations, and whether it actually sets up a correspondence between certain finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ and $U_t({}^L\widehat{\mathfrak{g}})$.

6.3. **The limit** $q \to 1$**.** Finally, let us consider the special case $q = 1$ of our formula for $T_1(z)$. We claim that it is closely related to the Bethe Ansatz formula for the first fundamental representation of $U_t(\widehat{\mathfrak{g}}^\vee)$, where $\widehat{\mathfrak{g}}^\vee = ^L(\widehat{^L\mathfrak{g}})$ (e.g., $(B_\ell^{(1)})^\vee = D_{\ell+1}^{(2)}$, and $(C_\ell^{(1)})^\vee = A_{2\ell-1}^{(2)}$). The latter can be found in [38, 30]. The only difference is that in the models associated to twisted affine algebras, Bethe Ansatz formulas contain terms of the form $Y_i(\epsilon^k z t^a)$. Our formulas coincide with formulas for the eigenvalues modulo the ϵ factors, which are absent in our formulas when $t = 1$ (these factors do appear in the limit $q \to \epsilon$, see the previous subsection). This indicates that our formulas at $q = 1$ are t–characters of finite-dimensional representations of some algebra closely related to $U_t(\widehat{\mathfrak{g}}^\vee)$.

It is again interesting to analize explicitly how finite-dimensional representations of the dual pairs of affine algebras, $\widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}}^\vee$, get connected in the deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$. This is clear for simply-laced $\mathfrak{g}$, but one can see rather intriguing effects for nonsimply-laced $\mathfrak{g}$, similar to those considered in the previous subsection.

For instance, in the case $\mathfrak{g} = B_\ell$ the deformed $\mathcal{W}$–algebras apparently interpolates between finite-dimensional representations of $B_\ell^{(1)}$ and $D_{\ell+1}^{(2)}$. The first fundamental representation of $B_\ell^{(1)}$ has dimension $2\ell + 1$, while the first fundamental representation of $D_{\ell+1}^{(2)}$ has dimension $2\ell + 2$. We see from the formula for $T_1(z)$ above that at $t = 1$ there are indeed $2\ell + 1$ terms corresponding to the weight spaces in the fundamental representation of $B_\ell^{(1)}$. On the other hand, at $q = 1$, one of the terms gets doubled, and there appear $2\ell + 2$ terms, corresponding to the weight spaces in the fundamental representation of $D_{\ell+1}^{(2)}$.

Similar effect probably occurs for the dual pairs $(F_4^{(1)}, E_6^{(2)})$ (26– and 27–dimensional fundamental representations, respectively) and $(G_2^{(1)}, D_4^{(3)})$ (7– and 8–dimensional fundamental representations, respectively).

In the case $\mathfrak{g} = C_\ell$ our formula for $T_1(z)$ connects the 2ℓ–dimensional fundamental representation of $C_\ell^{(1)}$ and the 2ℓ–dimensional fundamental representation of $A_{2\ell-1}^{(2)}$.

7. The case of $A_{2\ell}^{(2)}$

Up to now, we have not discussed the series $A_{2\ell}^{(2)}$ of self-dual twisted affine algebras. In this section we define the deformed $\mathcal{W}$–algebra associated to $A_{2\ell}^{(2)}$ and its free field realization. One may also view it as the $\mathcal{W}$–algebra associated to the non-reduced root system BC_ℓ. The results of this section generalize the results of Brazhnikov and Lukyanov [7], who constructed the deformed $\mathcal{W}$–algebra associated to $A_2^{(2)}$.

First we define the $\ell \times \ell$ symmetric matrix $B(q,t)$ associated to $A_{2\ell}^{(2)}$ as follows:

$$\begin{aligned} B_{ij}(q,t) &= [2]_q B_{ij}^A(q^2,t), \qquad (i,j) \neq (\ell,\ell), \\ B_{\ell\ell}(q,t) &= [2]_q (q^2 t^{-1} - 1 + q^{-2} t), \end{aligned}$$

where $(B_{ij}^A(q,t))$ is the B–matrix associated to A_ℓ.

By definition, the Heisenberg algebra $\mathcal{H}_{q,t}(A_{2\ell}^{(2)})$ has generators $a_i[n], i = 1, \dots, \ell$; $n \in \mathbb{Z}$, and relations (3.1) with $B(q,t)$ as above. We define the "dual" generators

$y_i[n]$ by formula (3.3) with $r_i = 2, \forall i$, and the operators e^{Q_j} by formula (3.6), where $B_{ij} = B_{ij}(1,1)$.

Now set

$$A_i(z) = t^2 q^{-4+2a_i[0]} : \exp\left(\sum_{m\neq 0} a_i[m] z^{-m}\right) :,$$

$$Y_i(z) = t^{i(2\ell+1-i)} q^{-2i(2\ell+1-i)+2y_i[0]} : \exp\left(\sum_{m\in\mathbb{Z}} y_i[m] z^{-m}\right) : .$$

The screening operators are defined by formulas (3.9) and (3.10), in which we set $r_i = 2, \forall i$.

Now we define the DCA $\mathbf{W}_{q,t}(A^{(2)}_{2\ell})$ as the maximal subalgebra of $(\mathbf{H}_{q,t}(A^{(2)}_{2\ell}), \pi_0)$, which commutes with the operators $S^-_i, i = 1, \dots, \ell$. We define the deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(A^{(2)}_{2\ell})$ as the associative algebra, topologically generated by the Fourier coefficients of fields from $\mathbf{W}_{q,t}(A^{(2)}_{2\ell})$. We expect an analogue of Conjecture 1 to hold in the $A^{(2)}_{2\ell}$ case.

Here is the explicit formula for the field $T_1(z)$:

$$T_1(z) = \sum_{i\in J} \Lambda_i(z),$$

where $J = \{1, \dots, \ell, 0, \overline{\ell}, \dots, \overline{1}\}$ and

$$\begin{aligned}
\Lambda_i(z) &=: Y_i(zq^{-2(i+1)}t^{i-1}) Y_{i-1}(zq^{-2i}t^i)^{-1} :, \qquad i = 1, \dots, \ell,\\
\Lambda_0(z) &= \frac{1-q^{-2}t^{-1}}{1-q^2t^{-1}} : Y_\ell(zq^{-2\ell}t^\ell) Y_\ell(zq^{-2(\ell+1)}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{i}}(z) &= q^{-4} : Y_{i-1}(zq^{-2(2\ell-i+1)}t^{2\ell-i+1}) Y_i(zq^{-2(2\ell-i+2)}t^{2\ell-i+2})^{-1} :, \qquad i = 1, \dots, \ell-1.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\Lambda_1(z) &= Y_1(z),\\
\Lambda_i(z) &=: \Lambda_{i-1}(z) A_{i-1}(zq^{-2(i-1)}t^{i-1})^{-1} :, \qquad i = 2, \dots, \ell,\\
\Lambda_0(z) &= \frac{1-q^{-2}t^{-1}}{1-q^2t^{-1}} : \Lambda_\ell(z) A_\ell(zq^{-2\ell}t^\ell)^{-1} :,\\
\Lambda_{\overline{\ell}}(z) &= \frac{1-q^{-2}t}{1-q^2t} : \Lambda_0(z) A_\ell(zq^{-2(\ell+1)}t^{\ell+1})^{-1} :,\\
\Lambda_{\overline{i}}(z) &=: \Lambda_{\overline{i+1}}(z) A_i(zq^{-2(2\ell-i+1)}t^{2\ell-i+1})^{-1}, \qquad i = 1, \dots, \ell-1.
\end{aligned}$$

Again, we find a perfect agreement between the specialization of this formula to $q = i$ and the formula obtained by analytic Bethe Ansatz in the $A^{(2)}_{2\ell}$ integrable model (see [38, 30])[1], in agreement with our general scheme outlined above.

Note however, that in contrast with the general case, $T_1(z)$ (and probably other fields as well) does not commute with the second set of screening operators, S^+_i.

[1]in the case of $A^{(2)}_2$ this has been observed in [7]

8. The scaling limit

8.1. **The scaling limit of the exchange relations and factorized scattering.** In this subsection we show that in the scaling limit the function $S_{ij}(x) = S_{Y_i,Y_j}(x)$ given by formula (3.15) becomes the S–matrix of the ith and jth particles of the affine Toda field theory associated to $\widehat{\mathfrak{g}}$ [6, 9, 8], in the integral form given by T. Oota [36].

Let us set

$$z = e^{-i\theta_1\epsilon}, \quad w = e^{-i\theta_2\epsilon},$$

$$q = \exp\frac{B}{2r^\vee h^\vee}\pi\epsilon, \qquad t = \exp\frac{B-2}{2h}\pi\epsilon.$$

Put $\theta = \theta_1 - \theta_2$. Then in the limit $\epsilon \to 0$ with θ_i and B, the function $S_{ij}(x)$ becomes:

$$S_{ij}(\theta) = \exp\left(-4\int_{-\infty}^{\infty}\frac{dk}{k}e^{ik\theta}\cdot\sinh\frac{2-B}{2h}\pi k\cdot\sinh\frac{B}{2r^\vee h^\vee}\pi k\cdot M_{ij}\left(e^{\frac{B}{2r^\vee h^\vee}\pi k}, e^{\frac{B-2}{2h}\pi k}\right)\right),$$

(the integral should be understood as the principal value).

This formula coincides with Oota's integral formula for the S–matrix of the affine Toda field theory [36].[2] This means that the scaling limit of our deformed $\mathcal{W}$–algebra $\mathcal{W}_{q,t}(\mathfrak{g})$ can be viewed as the Faddeev-Zamolodchikov algebra of the Toda theory. This has been suggested by Lukyanov in the A_ℓ case [32, 33] (see also [31]). The scaling limit of $\mathcal{W}_{q,t}(A_\ell)$ has also been studied in [23]. The scaling limit of our free field realization can be used to obtain explicit formulas for form-factors in general affine Toda field theories along the lines of [32, 33, 7].

On the other hand, the functions $S_{ij}(\theta)$ describe (at least in the simply-laced case) the scattering amplitudes for the lowest breather particles in the affine Toda field theories with imaginary coupling constant (these particles are bound states of solitons analogous to the breather particles of the sine-Gordon model). This fact is connected with the particle-breather duality of Toda theory [22, 26]. Note that the Toda theories with imaginary coupling constant are non-unitary (except for $\mathfrak{g} = A_1$). In order to have a physically meaningful quantum field theory, one has to restrict those theories to RSOS scattering states.

The functions (3.15) are elliptic generalizations of the S–matrices of the affine Toda field theories. Hence they can also be interpreted as the S–matrices of the RSOS model, whose scaling limit gives the corresponding restricted Toda theory with imaginary coupling constant. The algebra $W_{q,t}(\mathfrak{g})$ can therefore be interpreted as the Faddeev-Zamolodchikov algebra for this RSOS model (in the case $\mathfrak{g} = A_1$ this has been suggested in [32]). The embedding of $\mathcal{W}_{q,t}(\mathfrak{g})$ into $\mathcal{H}_{q,t}(\mathfrak{g})$ give us a bosonization of this Faddeev-Zamolodchikov algebra.

8.2. **The scaling limit of the screening currents.** Set

$$x = e^{2\pi i\tau u}, \qquad q = e^{2\pi i\tau}, \qquad t = e^{2\pi i\tau\beta}. \tag{8.1}$$

[2] Note that a factor of i is missing in front of the integral in [36], and that the matrix (C_{ab}) used in [36] is the transpose of the Cartan matrix according to the conventions of [27]. We thank T. Oota for clarifying this point to us.

Denote by $G^{\pm}_{ij}(w/z)$ be the function appearing in the quadratic relation between $S^{\pm}_i(z)$ and $S^{\pm}_j(w)$ from Sect. 3.5:

$$S^{\pm}_i(z)S^{\pm}_j(w) = G^{\pm}_{ij}\left(\frac{w}{z}\right) S^{\pm}_j(w)S^{\pm}_i(z).$$

Consider the limit of $G^{\pm}_{ij}(x)$ as $\tau \to +i0$ with x, q, and t given by (8.1). Using the modular properties of the function $\theta(z;a)$, we find:

$$\lim_{\tau\to+i0} \frac{\theta(e^{2\pi i\tau(u+\alpha)}; e^{2\pi i\tau})}{\theta(e^{2\pi i\tau(u+\gamma)}; e^{2\pi i\tau})} = \frac{\sin\pi(u+\alpha)}{\sin\pi(u+\gamma)}.$$

Applying this formula, we obtain the following limits for the functions $G^{-}_{ij}(x)$:

$$G^{-}_{kk}(x) \to \frac{\sin\frac{\pi}{2\beta}(u-2)}{\sin\frac{\pi}{2\beta}(u+2)}, \tag{8.2}$$

$$G^{-}_{km}(x) \to \frac{\sin\frac{\pi}{2\beta}(u+\beta-B_{km})}{\sin\frac{\pi}{2\beta}(u+\beta+B_{km})}, \qquad k \neq m. \tag{8.3}$$

By shifting the variables $u_k, k = 1, \ldots, \ell$, in the obvious way, we obtain the following relations between the scaling limits $s^{-}_k(u)$ of the screening currents:

$$s^{-}_k(u)s^{-}_m(v) = \frac{\sin\frac{\pi}{2\beta}(v-u-B_{km})}{\sin\frac{\pi}{2\beta}(v-u+B_{km})} s^{-}_m(v)s^{-}_k(u). \tag{8.4}$$

If we set $z = e^{\pi iu/\beta}, w = e^{\pi iv/\beta}$, and $q = e^{\pi i/\beta}$, then the function g^{-}_{km} appearing in the right hand side of formula (3.10) becomes

$$g^{-}_{km} = \frac{zq^{B_{ij}} - w}{z - wq^{B_{ij}}},$$

and we see that relations (8.4) coincide with the Drinfeld relations [11] in the subalgebra $U_q(\widehat{\mathfrak{n}})$ of $U_q(\widehat{\mathfrak{g}})$.

Analogously, we obtain the following limits for the functions $G^{+}_{km}(x)$:

$$G^{+}_{kk}(x) \to \frac{\sin\frac{\pi}{2r_{kk}}(u-2\beta)}{\sin\frac{\pi}{2r_{kk}}(u+2\beta)}, \tag{8.5}$$

$$G^{+}_{km}(x) \to \frac{\sin\frac{\pi}{2r_{km}}(u+r_{km}+\beta)}{\sin\frac{\pi}{2r_{km}}(u+r_{km}-\beta)}, \qquad k \neq m, B_{km} \neq 0. \tag{8.6}$$

By shifting the variables $u_k, k = 1, \ldots, \ell$, we obtain the following relations between the scaling limits $s^{+}_k(u)$ of the screening currents:

$$s^{+}_k(u)s^{+}_k(v) = \frac{\sin\frac{\pi}{2r_{ii}}(v-u-2\beta)}{\sin\frac{\pi}{2r_{ii}}(v-u+2\beta)} s^{+}_k(v)s^{+}_k(u), \tag{8.7}$$

$$s^{+}_k(u)s^{+}_m(v) = \frac{\sin\frac{\pi}{r_{km}}(v-u+\beta)}{\sin\frac{\pi}{r_{km}}(v-u-\beta)} s^{+}_m(v)s^{+}_k(u), \qquad k \neq m, B_{km} \neq 0. \tag{8.8}$$

Denote by g^+_{ij} the function appearing in the right hand sides of formulas (8.7), (8.8). Set $z = e^{\pi iu/r^\vee}, w = e^{\pi iv/r^\vee}$, $t = e^{-\pi i\beta/r^\vee}$. Then the functions g^+_{km} become

$$g^+_{kk} = \frac{zt^2 - w}{z - wt^2}, \qquad k \in R_l,$$

$$g^+_{km} = \frac{zt^{-1} - w}{z - wt^{-1}}, \qquad k \neq m \in R_l, B_{km} \neq 0,$$

$$g^+_{kk} = \frac{z^{r^\vee} t^{2r^\vee} - w^{r^\vee}}{z^{r^\vee} - w^{2r^\vee} t^{2r^\vee}}, \qquad k \in R_s,$$

$$g^+_{km} = \frac{z^{r^\vee} t^{-r^\vee} - w^{r^\vee}}{z^{r^\vee} - w^{r^\vee} t^{-r^\vee}}, \qquad k \neq m, k \in R_s, B_{km} \neq 0,$$

where R_l and R_s denote the sets of long and short simple roots, respectively. It is easy to recognize in these formulas the Drinfeld relations in the subalgebra $U_t(\widehat{\mathfrak{n}})$ of $U_t({}^L\widehat{\mathfrak{g}})$.

Thus, the relations of Sect. 3.5 between the screening currents $S^-_i(z)$ (resp., $S^+_i(z)$) can be considered as elliptic analogues of Drinfeld relations in $U_q(\widehat{\mathfrak{g}})$ (resp., $U_t({}^L\widehat{\mathfrak{g}})$).

9. Vertex operators

Vertex operators are constructed from the fields $Y_i(z)$ in the same way as the screening currents are constructed from the fields $A_i(z)$.

Introduce the operators $e^{Q_{\omega_i}}, i = 1, \dots, \ell$, acting from π_μ to $\pi_{\mu+\beta\omega_i}$, which satisfy commutation relations

$$[a_i[n], e^{Q_{\omega_j}}] = r_i \delta_{ij} \beta \delta_{n,0} e^{Q_{\omega_j}}.$$

Let

$$v^+_i[m] = \frac{y_i[m]}{q^{mr_i} - q^{-mr_i}}, \quad m \neq 0, \qquad v^+_i[0] = y_i[0]/r_i, \tag{9.1}$$

$$v^-_i[m] = \frac{y_i[m]}{t^m - t^{-m}}, \quad m \neq 0, \qquad v^-_i[0] = y_i[0]/\beta. \tag{9.2}$$

Now define the *fundamental vertex operators* by the formulas

$$V^+_i(z) = e^{Q_{\omega_i}/r_i} z^{v^+_i[0]} : \exp\left(-\sum_{m\neq 0} v^+_i[m] z^{-m}\right) :, \tag{9.3}$$

$$V^-_i(z) = e^{-Q_{\omega_i}/\beta} z^{-v^-_i[0]} : \exp\left(\sum_{m\neq 0} v^-_i[m] z^{-m}\right) : . \tag{9.4}$$

They satisfy the difference equations:

$$V^+_i(zq^{r_i}) = t^{-2(\rho^\vee,\omega_i)} q^{2r^\vee(\rho,\omega_i)} : Y_i(z) V^+_i(zq^{-r_i}) :, \tag{9.5}$$

and

$$V^-_i(zt^{-1}) = t^{-2(\rho^\vee,\omega_i)} q^{2r^\vee(\rho,\omega_i)} : Y_i(z) V^-_i(zt) : . \tag{9.6}$$

For $\mathfrak{g} = A_1$ these operators have been constructed in [34, 35], and for $\mathfrak{g} = A_\ell$ they have been constructed in [1].

In the conformal limit, $q \to 1, t = q^\beta$, the fields $V_i^-(z)$ and $V_i^+(z)$ become the bosonizations of the highest weight components of primary fields of $\mathcal{W}_\beta(\mathfrak{g})$. These primary fields correspond to the fundamental representations of $\mathfrak{g}$ (generalizations of $\Phi_{1,2}(z)$) and fundamental representations of ${}^L\mathfrak{g}$ (generalizations of $\Phi_{2,1}(z)$), respectively. Other components of these primary fields, corresponding to other weight components of the fundamental representations, are obtained by applying to them the screening operators of the same type ($-$ or $+$). Langlands duality interchanges the $-$ and $+$ vertex operators.

In the important works [34, 35, 1] it was shown that similar picture holds in the deformed situation in the case $\mathfrak{g} = A_\ell$. In those papers a complete bosonization of the vertex operators of the SOS models associated to $A_\ell^{(1)}$ and their restricted counterparts had been obtained. The highest weight components of those vertex operators are $V_i^-(z)$ (in our notation), and the rest are obtained by applying to them the deformed screening operators S_j^-. These are the "type I operators". It was explained in [1] that these operators are bosonizations of the half-infinite transfer-matrices on the lattice for the $A_\ell^{(1)}$ face model. We expect that the fields $V_i^-(z)$ associated to general Lie algebras can be used in the same fashion to obtain the bosonization of type I operators in the corresponding SOS and RSOS models [25].

The "type II operators" should be obtained in the same way, starting from $V_i^+(z)$. However, note that in the conformal limit the components of the type II operators correspond to weight spaces in the fundamental representations of ${}^L\mathfrak{g}$, rather than $\mathfrak{g}$.

Remark 3. Note that in the classical limits $t \to 1$ (respectively, $q \to \epsilon$), the vertex operators $V_i^+(z)$ (respectively, $V^-(z)$) become Baxter's Q–operators of the corresponding integrable model. □

Appendix A. Poisson algebras $\mathcal{W}_{q,1}(\mathfrak{g})$

In this section we make explicit computations in the Poisson algebra $\mathcal{W}_{q,1}(\mathfrak{g})$, when $\mathfrak{g}$ is a simple Lie algebra of classical type.

Recall that $M(q,t) = D(q,t)B(q,t)^{-1}D(q,t)$. For classical Lie algebras, these matrices are given in Appendix C. Set

$$\mathcal{B}_{ij}(x) = \sum_{m\in\mathbb{Z}} (q^m - q^{-m}) B_{ij}(q^m, 1) x^m,$$
$$\mathcal{M}_{ij}(x) = \sum_{m\in\mathbb{Z}} (q^m - q^{-m}) M_{ij}(q^m, 1) x^m.$$

We have the following Poisson brackets:

$$\{A_i(z), A_j(w)\} = \mathcal{B}_{ij}\left(\frac{w}{z}\right) A_i(z)A_j(w),$$
$$\{Y_i(z), Y_j(w)\} = \mathcal{M}_{ij}\left(\frac{w}{z}\right) Y_i(z)Y_j(w).$$

A.1. A_ℓ case.

In this section we recall the results of [17].

We have:

$$\Lambda_i(z) = Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \qquad i = 1, \dots, \ell+1.$$

The generators of $\mathcal{W}_{q,1}(A_\ell)$ are

$$T_i(z) = \sum_{1\le j_1<\ldots<j_i\le n+1} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2)\ldots\Lambda_{j_{i-1}}(zq^{2(i-2)})\Lambda_{j_i}(zq^{2(i-1)}), \quad i=1,\ldots,\ell+1.$$

We have: $T_{\ell+1}(z)=1$.

The Poisson brackets are [17]:

$$\begin{aligned}\{T_i(z),T_j(w)\} &= \mathcal{M}_{ij}\left(\frac{wq^{j-i}}{z}\right)T_i(z)T_j(w)\\ &+\sum_{p=1}^{i}\delta\left(\frac{w}{zq^{2p}}\right)T_{i-p}(w)T_{j+p}(z)\\ &-\sum_{p=1}^{i}\delta\left(\frac{wq^{2(j-i+p)}}{z}\right)T_{i-p}(z)T_{j+p}(w),\end{aligned}$$

if $i\le j$ and $i+j\le \ell+1$; and

$$\begin{aligned}\{T_i(z),T_j(w)\} &= \mathcal{M}_{ij}\left(\frac{wq^{j-i}}{z}\right)T_i(z)T_j(w)\\ &+\sum_{p=1}^{N-j}\delta\left(\frac{w}{zq^{2p}}\right)T_{i-p}(w)T_{j+p}(z)\\ &-\sum_{p=1}^{N-j}\delta\left(\frac{wq^{2(j-i+p)}}{z}\right)T_{i-p}(z)T_{j+p}(w),\end{aligned}$$

if $i\le j$ and $i+j>\ell+1$.

A.2. B_ℓ **case.** This and the next two subsections are taken from [18].

We have:

$$\begin{aligned}\Lambda_i(z) &= Y_i(zq^{-2i+2})Y_{i-1}(zq^{-2i})^{-1}, \qquad i=1,\ldots,\ell-1,\\ \Lambda_\ell(z) &= Y_\ell(zq^{-2\ell+3})Y_\ell(zq^{-2\ell+1})Y_{\ell-1}(zq^{-2\ell}t^\ell)^{-1},\\ \Lambda_0(z) &= Y_\ell(zq^{-2\ell+3})Y_\ell(zq^{-2\ell-1})^{1},\\ \Lambda_{\bar\ell}(z) &= Y_{\ell-1}(zq^{-2\ell+2})Y_\ell(zq^{-2\ell+1})^{-1}Y_\ell(zq^{-2\ell-1})^{-1},\\ \Lambda_{\bar i}(z) &= Y_{i-1}(zq^{-4\ell+2i+2})Y_i(zq^{-4\ell+2i})^{-1}, \qquad i=1,\ldots,\ell-1.\end{aligned}$$

Let

$$T_i(z) = \sum_{\{j_1,\ldots,j_i\}\in S}\Lambda_{j_1}(z)\Lambda_{j_2}(zq^4)\ldots\Lambda_{j_i}(zq^{4i-4}), \qquad i=1,\ldots,\ell-1,$$

where S is the set of $\{j_1,\ldots,j_i\}$, such that $j_\alpha<j_{\alpha+1}$ or $j_\alpha=j_{\alpha+1}=\ell+1, \alpha = 1,\ldots,i-1$.

The formula for $T_\ell(z)$ can be found in [17].

Remark 4. In the formulas above, q corresponds to $q^{1/2}$ of [17]. □

The Poisson brackets are:

$$
\begin{aligned}
\{\Lambda_i(z),\Lambda_i(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_i(w),\\
\{\Lambda_i(z),\Lambda_j(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w)+\left(\delta\left(\frac{w}{zq^4}\right)-\delta\left(\frac{w}{z}\right)\right.\\
&\qquad \left.+\,\delta\left(\frac{w}{zq^{4\ell-4i+2}}\right)\delta_{i,\bar{j}}-\delta\left(\frac{w}{zq^{4\ell-4i-2}}\right)\delta_{i,\bar{j}}\right)\Lambda_i(z)\Lambda_j(w),
\end{aligned}
$$

if $i<j$.

These formulas imply for $\ell>2$:

$$
\begin{aligned}
\{T_1(z),T_1(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)T_1(z)T_1(w)\\
&\quad+\delta\left(\frac{w}{zq^4}\right)T_2(z)-\delta\left(\frac{wq^4}{z}\right)T_2(w)\\
&\quad+\delta\left(\frac{w}{zq^{4\ell-2}}\right)-\delta\left(\frac{w^{4\ell-2}}{z}\right).
\end{aligned}
$$

In the case of B_2 the formulas for the Poisson brackets can be read off the formulas for the Poisson brackets in the case of C_2 given in Sect. A.4.

A.3. C_ℓ **case.** We have:

$$
\begin{aligned}
\Lambda_i(z) &= Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \qquad i=1,\dots,\ell,\\
\Lambda_{\bar{i}}(z) &= Y_{i-1}(zq^{-2\ell+i-2})Y_i(zq^{-2\ell+i-3})^{-1}, \qquad i=1,\dots,\ell.
\end{aligned}
$$

Let

$$
T_i(z)=\sum_{\{j_1,\dots,j_i\}\in S}\Lambda_{j_1}(z)\Lambda_{j_2}(zq)\dots\Lambda_{j_i}(zq^{i-1}), \qquad i=1,\dots,\ell,
$$

where S is the set of $\{j_1,\dots,j_i\}$, such that $1\le j_1<\dots<j_i\le 2\ell$ and if $j_\alpha=k, j_\beta=2\ell+1-k$ for some $k=1,\dots,\ell$, then $l\le \ell+\alpha-\beta$.

Remark 5. In the formulas above, q corresponds to $q^{1/2}$ of [17]. □

The Poisson brackets are:

$$
\begin{aligned}
\{\Lambda_i(z),\Lambda_i(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_i(w),\\
\{\Lambda_i(z),\Lambda_j(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w)+\left(\delta\left(\frac{w}{zq^2}\right)-\delta\left(\frac{w}{z}\right)\right.\\
&\qquad \left.+\,\delta\left(\frac{w}{zq^{2\ell-2i+4}}\right)\delta_{i,\bar{j}}-\delta\left(\frac{w}{zq^{2\ell-2i+2}}\right)\delta_{i,\bar{j}}\right)\Lambda_i(z)\Lambda_j(w),
\end{aligned}
$$

if $i<j$.

These formulas imply:

$$\begin{aligned}\{T_1(z), T_1(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right) T_1(z)T_1(w) \\ &+ \delta\left(\frac{w}{zq^2}\right) T_2(z) - \delta\left(\frac{wq^2}{z}\right) T_2(w) \\ &+ \delta\left(\frac{w}{zq^{2\ell+2}}\right) - \delta\left(\frac{w^{2\ell+2}}{z}\right).\end{aligned}$$

A.4. **Poisson brackets for C_2.** In this subsection we give a compute description of the Poisson algebra $\mathcal{W}_{q,1}(C_2)$.

$$\begin{aligned}\{T_1(z), T_1(w)\} &= \sum_{m\in\mathbb{Z}} \frac{(q^m - q^{-m})(q^{2m}+q^{-2m})}{q^{3m}+q^{-3m}} \left(\frac{w}{z}\right)^m T_1(z)T_1(w) \\ &+ \delta\left(\frac{w}{zq^2}\right) T_2(z) - \delta\left(\frac{wq^2}{z}\right) T_2(w) \\ &+ \delta\left(\frac{w}{zq^6}\right) - \delta\left(\frac{wq^6}{z}\right). \\ \{T_1(z), T_2(w)\} &= \sum_{m\in\mathbb{Z}} \frac{q^{2m}-q^{-2m}}{q^{3m}+q^{-3m}} \left(\frac{wq}{z}\right)^m T_1(z)T_2(w) \\ &+ \delta\left(\frac{w}{zq^4}\right) T_1(w) - \delta\left(\frac{wq^6}{z}\right) T_1(wq^2), \\ \{T_2(z), T_2(w)\} &= \sum_{m\in\mathbb{Z}} \frac{(q^{2m}-q^{-2m})(q^m+q^{-m})}{q^{3m}+q^{-3m}} \left(\frac{w}{z}\right)^m T_2(z)T_2(w) \\ &+ \delta\left(\frac{w}{zq^4}\right) \big(T_1(z)T_1(zq^2) - T_2(zq^2)\big) \\ &- \delta\left(\frac{wq^4}{z}\right) \big(T_1(w)T_1(wq^2) - T_2(wq^2)\big) \\ &+ \delta\left(\frac{w}{zq^6}\right) - \delta\left(\frac{wq^6}{z}\right).\end{aligned}$$

A.5. D_ℓ **case.** We have:

$$\begin{aligned}\Lambda_i(z) &= Y_i(zq^{-i+1})Y_{i-1}(zq^{-i})^{-1}, \qquad i = 1,\dots,\ell-2, \\ \Lambda_{\ell-1}(z) &= Y_\ell(zq^{-\ell+2})Y_{\ell-1}(zq^{-\ell+2})Y_{\ell-2}(zq^{-\ell+1})^{-1}, \\ \Lambda_\ell(z) &= Y_{\ell-1}(zq^{-\ell+2})Y_\ell(zq^{-\ell})^{-1}, \\ \Lambda_{\overline{\ell}}(z) &= Y_\ell(zq^{-\ell+2})Y_{\ell-1}(zq^{-\ell})^{-1}, \\ \Lambda_{\overline{\ell-1}}(z) &= Y_{\ell-2}(zq^{-\ell+1})Y_{\ell-1}(zq^{-\ell})^{-1}Y_\ell(zq^{-\ell})^{-1}, \\ \Lambda_{\overline{i}}(z) &= Y_{i-1}(zq^{-2\ell+i+2})Y_i(zq^{-2\ell+i+1})^{-1}.\end{aligned}$$

Let

$$T_i(z) = \sum_{\{j_1,\dots,j_i\}\in S} \Lambda_{j_1}(z)\Lambda_{j_2}(zq^2)\dots\Lambda_{j_i}(zq^{2i-2}), \qquad i = 1,\dots,\ell-2,$$

where S is the set of $\{j_1,\dots,j_i\}$, such that $j_\alpha < j_{\alpha+1}$ or $j_\alpha = j_{\alpha+1}+1 = \ell+1, \alpha = 1,\dots,i-1$.

The formulas for $T_{\ell-1}(z)$ and $T_\ell(z)$, which correspond to the spinor representations of $D_\ell^{(1)}$ are given in [17].

The following formulas are due to A. Kogan [28].

$$\begin{aligned}
\{\Lambda_i(z),\Lambda_i(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_i(w),\\
\{\Lambda_i(z),\Lambda_j(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)\Lambda_i(z)\Lambda_j(w) + \left(\delta\left(\frac{w}{zq^2}\right) - \delta\left(\frac{w}{z}\right)\right.\\
&\qquad \left. + \,\delta\left(\frac{w}{zq^{2\ell-2i}}\right)\delta_{i,\bar{j}} - \delta\left(\frac{w}{zq^{2\ell-2i-2}}\right)\delta_{i,\bar{j}}\right)\Lambda_i(z)\Lambda_j(w),
\end{aligned}$$

if $i<j$.

$$\begin{aligned}
\{T_1(z),T_1(w)\} &= \mathcal{M}_{11}\left(\frac{w}{z}\right)T_1(z)T_1(w)\\
&\quad + \delta\left(\frac{w}{zq^2}\right)T_2(z) - \delta\left(\frac{wq^2}{z}\right)T_2(w)\\
&\quad + \delta\left(\frac{w}{zq^{2\ell-2}}\right) - \delta\left(\frac{w^{2\ell-2}}{z}\right).
\end{aligned}$$

Appendix B. The difference version of the Drinfeld-Sokolov reduction in the case $\mathfrak{g} = C_2$

B.1. Recollections. Let us briefly recall the difference Drinfeld-Sokolov reduction scheme [21, 40]. Let G be a simple algebraic group, and $\mathcal{G} = G((z))$ be the corresponding formal loop group. Consider the following action of $\mathcal{G}$ on itself by difference gauge transformations:

$$g(z)\cdot x(z) = g(zp)x(z)g(z)^{-1}, \tag{B.1}$$

where p is a non-zero complex number. Let $s_i, i=1,\dots,\ell$, be the simple reflections from the Weyl group $W = N(H)/H$, where H is the Cartan subgroup of G, and $N(H)$ is its normalizer. Choose once and for all their lifts $n_i, i=1,\dots,\ell$, to $N(H)\subset G$. Denote by U the upper unipotent subgroup of G, and by U_i its one-dimensional unipotent subgroup corresponding to the ith simple root. Now set

$$M = Un_1\dots n_\ell U, \qquad C = U_1 n_1 U_1 n_2 \dots U_\ell n_\ell.$$

Let $\mathcal{U} = U((z)), \mathcal{M} = M((z)), \mathcal{C} = C((z))$.

Theorem 4 ([40])**.** *The restriction of the action (B.1) to $\mathcal{U}\subset\mathcal{G}$ preserves $\mathcal{M}$. The resulting action of $\mathcal{U}$ on $\mathcal{M}$ is free, and $\mathcal{C}$ is the cross-section of this action.*

Furthermore, in [21] (for $\mathfrak{g} = A_1$) and [40] (in general) a Poisson structure was defined on $\mathcal{G}$, which descends to a well-defined Poisson structure on $\mathcal{M}/\mathcal{U} \simeq \mathcal{C}$. The corresponding Poisson algebra of functions on $\mathcal{C}$ is the Poisson algebra $\mathcal{W}^p(\mathfrak{g})$ considered in Sect. 4.3.

Let $\mathcal{H}$ be the formal loop group of H. We define following [40] the difference Miura transformation $m : \mathcal{H} \to \mathcal{C}$. Let $\overline{U}$ be the lower unipotent subgroup of G. There is a unique element $f \in M \cap \overline{U}$. Denote by s the Coxeter element of W: $s = s_1 \ldots s_\ell$ (note that s here corresponds to s^{-1} of [40]). Define an embedding $i : \mathcal{H} \to \mathcal{M}$ that sends $x \in \mathcal{H}$ to $x \cdot f \cdot s^{-1}(x^{-1})$. It is easy to see that $i(x) \in \mathcal{M}$.

The difference Miura transformation $m : \mathcal{H} \to \mathcal{C}$ is by definition the composition of i and the projection $\mathcal{M} \to \mathcal{C}$. In [40] a Poisson structure was defined on $\mathcal{H}$, that makes the map m Poisson.

The above construction has been worked out explicitly in the case $\mathfrak{g} = A_\ell$ in [21, 40]: it was shown that the Poisson algebra $\mathcal{W}^t(A_\ell)$ is isomorphic to the Poisson algebra $\mathcal{W}_{t,1}(A_\ell)$, and the Miura transformations for the two Poisson algebras coincide. This proves Conjecture 3 in the case $\mathfrak{g} = A_\ell$. In the next subsection we prove Conjecture 3 in the case $\mathfrak{g} = C_2$.

B.2. The case of C_2. We realize $C_2 = \mathrm{Sp}_4$ as the group of 4×4 matrices A with $\det A = 1$, which satisfy

$$A^t J A = J,$$

where

$$J = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

We choose

$$n_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad n_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have

$$U = \left\{ \begin{pmatrix} 1 & a & b + \frac{ad}{2} & c \\ 0 & 1 & d & b - \frac{ad}{2} \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}, a, b, c, d \in \mathbb{C} \right\},$$

$$U_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \qquad U_2 = \left\{ \begin{pmatrix} 1 & \beta & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\beta \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and so

$$C = \left\{ \begin{pmatrix} S_1 & 1 & 0 & 0 \\ -S_2 & 0 & S_1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, S_1, S_2 \in \mathbb{C} \right\}. \tag{B.2}$$

Next, we choose:

$$f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix},$$

and write a typical element of H as

$$x = \begin{pmatrix} x_2^{1/2} & 0 & 0 & 0 \\ 0 & x_1 x_2^{-1/2} & 0 & 0 \\ 0 & 0 & x_1^{-1} x_2^{1/2} & 0 \\ 0 & 0 & 0 & x_2^{-1/2} \end{pmatrix}.$$

Now, to find explicit formulas for the difference Miura transformation, we have to represent the matrix (B.2) as

$$u(zt) \cdot x(z) \cdot f \cdot s^{-1}(x(z)^{-1}) \cdot u(z)^{-1},$$

where $u(z) \in \mathcal{U}$. Straightforward but tedious calculation gives:

$$\begin{aligned} S_1(z) = {} & x_1(z) + x_1(zp)^{-1} x_1(z)^2 x_2(z)^{-1} + x_1(zp)^{-1} x_2(zp) \\ & + x_1(zp^2)^{-1}, \end{aligned} \tag{B.3}$$

$$\begin{aligned} S_2(z) = {} & x_1(z)^2 x_2(z)^{-1} + x_2(zp) + 2 x_1(zp) x_1(zp^2)^{-1} \\ & + x_1(zp)^2 x_1(zp^2)^{-2} x_2(zp)^{-1} + x_1(zp^2)^{-2} x_2(zp^2). \end{aligned} \tag{B.4}$$

Now set $p = t^2$. If we let $Y_1(z) = x_1(z)$, and $Y_2(z) = x_1(zt^{-1})^2 x_2(zt^{-1})^{-1}$, then formula (B.3) becomes:

$$S_1(z) = Y_1(z) + Y_2(zt) Y_1(zt^2)^{-1} + Y_2(zt^3)^{-1} Y_1(zt^2) + Y_1(zt^4)^{-1}. \tag{B.5}$$

It coincides with the formula for the generator $T_1(z)$ of $\mathcal{W}_{1,t}(C_2)$.

On the other hand, if we let $\widetilde{Y}_1(z) = x_1(z)^2 x_2(z)^{-1}$, and $\widetilde{Y}_2(z) = x_1(zt)$, then formula (B.4) becomes:

$$\begin{aligned} S_2(z) = {} & \widetilde{Y}_1(z) + \widetilde{Y}_2(zt)^2 \widetilde{Y}_1(zt^2)^{-1} + 2\widetilde{Y}_2(zt) \widetilde{Y}_2(zt^3)^{-1} \\ & + \widetilde{Y}_2(zt^3)^{-2} \widetilde{Y}_1(zt^2) + \widetilde{Y}_1(zt^4)^{-2}. \end{aligned} \tag{B.6}$$

It coincides with the formula for the generator $T_1(z)$ of $\mathcal{W}_{1,t}(B_2)$, which is the same as the generator $T_2(z)$ of $\mathcal{W}_{1,t}(C_2)$ up to a shift. Thus, the generators of $\mathcal{W}_{1,t}(C_2)$ and $\mathcal{W}^{t^2}(C_2)$ coincide.

Now we compute the Poisson brackets between $Y_i(z)$ using formula (4.1) of [40]. First we find the eigenvectors of the transformation $x \to s^{-1}(x)$ on H. These are represented

by matrices

$$e_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

with the eigenvalues $-i$ and i, respectively. In terms of these elements, the Poisson tensor of [40] on $\mathcal{H}$ is represented by the formal power series

$$-\sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n (1-t^{2n})\frac{1}{2}\left(\frac{1-i}{1-it^{2n}} e_1 \otimes e_2 + \frac{1+i}{1+it^{2n}} e_2 \otimes e_1\right). \tag{B.7}$$

(recall that $p = t^2$).

Now let $B_i(z)$ be (i,i)th matrix entry of element of $\mathcal{H}$. Using formula (B.7), we obtain the following Poisson brackets:

$$\begin{aligned}\{B_1(z), B_1(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n (1-t^{2m})\frac{1}{2}\left(\frac{1-i}{1-it^{2n}} + \frac{1+i}{1+it^{2n}}\right) B_1(z)B_1(w) \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{t^{2n}-t^{-2n}}{t^{2n}+t^{-2n}} B_1(z)B_1(w),\end{aligned}$$

and similarly:

$$\begin{aligned}\{B_1(z), B_2(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{(t^n-t^{-n})^2}{t^{2n}+t^{-2n}} B_1(z)B_2(w), \\ \{B_2(z), B_2(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{t^{2n}-t^{-2n}}{t^{2n}+t^{-2n}} B_2(z)B_2(w),\end{aligned}$$

Under the map $x \to x{\cdot}s^{-1}(x^{-1})$, $x_1(z)$ goes to $B_1(z)$, and $x_1(z)x_2(z)^{-1}$ goes to $B_2(z)$. Hence $Y_1(z) = B_1(z), Y_2(z) = B_1(zt^{-1})B_2(zt^{-1})$, and straightforward calculation gives us the following Poisson brackets:

$$\begin{aligned}\{Y_1(z), Y_1(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{t^{2n}-t^{-2n}}{t^{2n}+t^{-2n}} Y_1(z)Y_1(w), \\ \{Y_1(z), Y_2(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{2(t^n-t^{-n})}{t^{2n}+t^{-2n}} Y_1(z)Y_2(w), \\ \{Y_2(z), Y_2(w)\} &= \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n \frac{2(t^{2n}-t^{-2n})}{t^{2n}+t^{-2n}} Y_2(z)Y_2(w).\end{aligned}$$

This agrees with the Poisson bracket

$$\{Y_i(z), Y_j(w)\} = \sum_{n\in\mathbb{Z}} \left(\frac{w}{z}\right)^n (t^n - t^{-n}) M_{C_2}(1, t^n)\; Y_i(z)Y_j(w),$$

in $\mathcal{H}_{1,t}(C_2)$. Hence $\mathcal{W}_{1,t}(C_2)$ and $\mathcal{W}^{t^2}(C_2)$ are isomorphic as Poisson algebras, which is what we wanted to show.

Appendix C. The matrices $M(q,t)$ for Lie algebras of classical types

A_ℓ series.

$$M_{ij}(q,t) = \frac{(q^i t^{-i} - q^{-i} t^i)(q^{\ell+1-j} t^{-\ell-1+j} - q^{-\ell-1+j} t^{\ell+1-j})}{q^{\ell+1} t^{-\ell-1} - q^{-\ell-1} t^{\ell+1}}, \qquad i \leq j.$$

B_ℓ series.

$$\begin{aligned}
M_{ij}(q,t) &= \frac{(q^{2i} t^{-i} - q^{-2i} t^i)(q^{2\ell-1-2j} t^{-\ell+j} + q^{-2\ell+1+2j} t^{\ell-i})(q+q^{-1})}{(q^2 t^{-1} - q^{-2} t)(q^{2\ell-1} t^{-\ell} + q^{-2\ell+1} t^{\ell})}, & 1 \leq i \leq j < \ell,\\
M_{i\ell}(q,t) &= \frac{(q^{2i} t^{-i} - q^{-2i} t^i)(q+q^{-1})}{(q^2 t^{-1} - q^{-2} t)(q^{2\ell-1} t^{-\ell} + q^{-2\ell+1} t^{\ell})}, & 1 \leq i < \ell,\\
M_{\ell\ell}(q,t) &= \frac{q^{2\ell} t^{-\ell} - q^{-2\ell} t^{\ell}}{(q^2 t^{-1} - q^{-2} t)(q^{2\ell-1} t^{-\ell} + q^{-2\ell+1} t^{\ell})}.
\end{aligned}$$

C_ℓ series.

$$M_{ij}(q,t) = \frac{(q^i t^{-i} - q^{-i} t^i)(q^{\ell+1-j} t^{-\ell+j} + q^{-\ell-1+j} t^{\ell-j})}{(q t^{-1} - q^{-1} t)(q^{\ell+1} t^{-\ell} + q^{-\ell-1} t^{\ell})}, \qquad 1 \leq i \leq j \leq \ell.$$

D_ℓ series.

$$\begin{aligned}
M_{ij}(q,t) &= \frac{(q^i t^{-i} - q^{-i} t^i)(q^{\ell-1-j} t^{-(\ell-1-j)} + q^{-(\ell-1-j)} t^{\ell-1-j})}{q^{\ell-1} t^{-\ell+1} + q^{-\ell+1} t^{\ell-1}}, & 1 \leq i \leq j \leq \ell-1,\\
M_{i\ell}(q,t) &= \frac{q^i t^{-i} - q^{-i} t^i}{q^{\ell-1} t^{-\ell+1} + q^{-\ell+1} t^{\ell-1}}, & 1 \leq i \leq \ell-2,\\
M_{\ell-1,\ell}(q,t) &= \frac{q^{\ell-2} t^{-\ell+2} - q^{-\ell+2} t^{\ell-2}}{(q t^{-1} + q^{-1} t)(q^{\ell-1} t^{-\ell+1} + q^{-\ell+1} t^{\ell-1})},\\
M_{\ell-1,\ell-1}(q,t) = M_{\ell\ell}(q,t) &= \frac{q^{\ell} t^{-\ell} - q^{-\ell} t^{\ell}}{(q t^{-1} + q^{-1} t)(q^{\ell-1} t^{-\ell+1} + q^{-\ell+1} t^{\ell-1})}.
\end{aligned}$$

Appendix D. Deformed Chiral Algebras

Definition. A deformed chiral algebra (DCA) with a representation is a collection of the following data:

- A vector space V called the space of fields.
- A vector space $W = \cup_{n \geq 0} W_n$ called the space of states, which is union of finite-dimensional subspaces W_n. We consider a topology on W in which $\{W_n\}_{n \geq 0}$ is the base of open neighborhoods of 0.

- A linear map $Y : V \to \operatorname{End} W\widehat{\otimes}[[z,z^{-1}]]$ that is for each $A \in V$, a formal power series $Y(A,z) = \sum_{n\in\mathbb{Z}} A_n z^{-n}$, where each A_n is a linear operator $W \to W$, such that $A_n \cdot W_m \subset W_{m+N(n)}, \forall m \geq 0$, for some $N(n) \in \mathbb{Z}$.
- A meromorphic function $S(x) : \mathbb{C}^\times \to \operatorname{Aut}(V \otimes V)$, satisfying the Yang-Baxter equation:
$$S_{12}(z)S_{13}(zw)S_{23}(w) = S_{23}(w)S_{13}(zw)S_{12}(z),$$
for all $z, w \in \mathbb{C}^\times$.
- A lattice $L \subset \mathbb{C}^\times$, which contains the poles of $S(x)$.
- A vector $\Omega \in V$, such that $Y(\Omega, z) = \mathrm{Id}$.

The data of deformed chiral algebra should satisfy the following axioms:

(1) For any $A_i \in V, i = 1, \ldots, n$, the composition $Y(A_1,z_1)\ldots Y(A_n,z_n)$ converges in the domain $|z_1| \gg \ldots \gg |z_n|$ and can be continued to a meromorphic operator valued function
$$R(Y(A_1,z_1)\ldots Y(A_n,z_n)) : (\mathbb{C}^\times)^n \to \operatorname{Hom}(W,\overline{W}),$$
where $\overline{W}$ is the completion of W with respect to its topology.

(2) Denote $R(Y(A,z)Y(B,w))$ by $Y(A\otimes B; z,w)$. Then
$$Y(A\otimes B; z,w) = Y(S(w/z)(B\otimes A); w,z).$$

(3) The poles of the meromorphic function $R(Y(A,z)Y(B,w))$ lie on the lines $z = w\gamma$ where $\gamma \in L$. For each such line and $n \geq 0$, there exists $C_n \in V$, such that
$$\operatorname{Res}_{z=w\gamma} R(Y(A,z)Y(B,w))(z-w\gamma)^n \frac{dz}{z} = Y(C_n, w).$$

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